Pro excision and h-descent for K-theory

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Abstract

In this paper it is proved that K-theory (and Hochschild and cyclic homology) satisfies pro versions of both excision for ideals (of commutative Noetherian rings) and descent in the h-topology in characteristic zero; this is achieved by passing to the limit over all infinitesimal thickenings of the ideal or exceptional fibre in question.

INTRODUCTION

Algebraic K-theory satisfies neither excision nor descent in the h-topology, in contrast to, e.g., Weibel's homotopy invariant K-theory [9], or periodic cyclic homology [4]. The aim of this paper is to show that K-theory does infact satisfy both excision and h-descent (for commutative Noetherian rings in characteristic zero) *when we pass to the limit over all thickenings of the ideal or exceptional fibre in question.* We also prove the same for both Hochschild and cyclic homology.

This process of passing to the limit is achieved either via homotopy limits or, even better, via the use of the category ProAb of formal pro abelian groups (or even via pro spectra). Before stating the main theorems, we remind the reader that a formal pro group is simply a formal projective system $A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots$ of abelian groups; such a system is commonly denoted "$\varprojlim$"$_r A_r$ or $\{A_r\}_r$, and ProAb is the abelian category of all such systems.

The following is our first main theorem:

Theorem 0.1 (Pro-excision). *Let $A \to B$ be an essentially finite type morphism of (commutative) Noetherian $\mathbb{Q}$-algebras, and let I be an ideal of A mapped isomorphically to an ideal of B. Then the natural morphism of relative pro groups*

$$\text{``}\varprojlim_r\text{''} \, K_n(A, I^r) \longrightarrow \text{``}\varprojlim_r\text{''} \, K_n(B, I^r)$$

is an isomorphism for all $n \in \mathbb{Z}$.

Therefore there is a long exact, Mayer–Vietoris sequence in ProAb

$$\cdots \longrightarrow K_n(A) \longrightarrow \text{``}\varprojlim_r\text{''} \, K_n(A/I^r) \oplus K_n(B) \longrightarrow \text{``}\varprojlim_r\text{''} \, K_n(B/I^r) \longrightarrow \cdots$$

The theorem is equivalent to the brief statement that the following diagram of pro spectra is homotopy cartesian:

$$
\begin{array}{ccc}
K(A) & \longrightarrow & K(B) \\
\downarrow & & \downarrow \\
\text{``}\varprojlim_r\text{''}\, K(A/I^r) & \longrightarrow & \text{``}\varprojlim_r\text{''}\, K(B/I^r)
\end{array}
$$

Since we do not intend to discuss model structures on the category of pro spectra, the reader should merely accept this as an alternate statement of the theorem, and should do the same with other similar squares of pro spectra which will appear; it implies that the diagram of spectra

$$
\begin{array}{ccc}
K(A) & \longrightarrow & K(B) \\
\downarrow & & \downarrow \\
\operatorname{holim}_r K(A/I^r) & \longrightarrow & \operatorname{holim}_r K(B/I^r)
\end{array}
$$

is homotopy cartesian. The theorem was proved in a special case by A. Krisha [13], and our method of proof mimics his approach.

Our second main theorem is the following:

Theorem 0.2 (Pro-h-descent)**.** *Let k be a field of characteristic zero, and let*

$$
\begin{array}{ccc}
Y' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
$$

be an abstract blow-up square of separated schemes of finite type over k. That is, the square is a pull-back, $Y \to X$ is a closed embedding, $X' \to X$ is proper, and $X' \setminus Y' \to X \setminus Y$ is an isomorphism. Then the natural morphism of relative pro groups

$$
\text{``}\varprojlim_r\text{''}\, K_n(X, rY) \longrightarrow \text{``}\varprojlim_r\text{''}\, K_n(X', rY')
$$

is an isomorphism for all $n \in \mathbb{Z}$, where rY denotes the rth infinitesimal thickening of Y inside X (and similarly for Y', X').

Just as in the case of excision, the second theorem implies that the squares

$$
\begin{array}{ccc}
K(X) & \longrightarrow & K(X') \\
\downarrow & & \downarrow \\
\text{``}\varprojlim_r\text{''}\, K(rY) & \longrightarrow & \text{``}\varprojlim_r\text{''}\, K(rY')
\end{array}
\qquad
\begin{array}{ccc}
K(X) & \longrightarrow & K(X') \\
\downarrow & & \downarrow \\
\operatorname{holim}_r K(rY) & \longrightarrow & \operatorname{holim}_r K(rY')
\end{array}
$$

are homotopy cartesian and that there are resulting long exact, Mayer–Vietoris sequences. If $X' \to X$ is finite then the second theorem reduces to the first.

It is a consequence of our methods that both theorems remain true for Hochschild and cyclic homology. In particular, if $A \to B$ is a morphism of k-algebras and I is an ideal of

A such that the conditions of theorem 0.1 are satisfied, then there is a long exact, Mayer–Vietoris sequence

$$\cdots \to HH_n^k(A) \to \text{``}\varprojlim_r\text{''} \, HH_n^k(A/I^r) \oplus HH_n^k(B) \to \text{``}\varprojlim_r\text{''} \, HH_n^k(B/I^r) \to \cdots$$

If $k \to B$ is 'virtually geometrically regular', a condition which is satisfied in most natural cases (see the appendix), then this even breaks into short exact sequences (see remark 4.3). It follows that

$$HH_n^k(A) \longrightarrow HH_n^k(A/I^r) \oplus HH_n(B)$$

is injective for $r \gg 0$, a result which we consider to be of independent interest.

The key to the proofs is a collection of Artin–Rees type properties for André–Quillen and Hochschild homologies. For example, if I is an ideal of a k-algebra A, then we give conditions for the natural maps

$$\text{``}\varprojlim_r\text{''} \, HH_n^k(A) \otimes_A A/I^r \longrightarrow \text{``}\varprojlim_r\text{''} \, HH_n^k(A/I^r)$$

to be isomorphisms.

An immediate corollary of such Artin–Rees results is the following pro Hochschild–Konstant–Rosenberg theorem (a version of which can also be found in [5]):

Theorem 0.3. *(See corollary A.8) Let $k \to A$ be a geometrically regular morphism of Noetherian rings, let I be an ideal of A, and let $n \geq 0$ be such that $n!$ is invertible in A. Then the natural map*

$$\text{``}\varprojlim_r\text{''} \, \Omega^n_{A/I^r|k} \longrightarrow \text{``}\varprojlim_r\text{''} \, HH_n^k(A/I^r)$$

is an isomorphism.

In the remainder of this introduction we will discuss the general problem of excision and descent, then offer some examples, and then outline the proofs.

Excision and descent

It has been known at least since work by R. Swan [25] that K-theory fails to satisfy excision; i.e., if $A \to B$ is a morphism of rings and I is an ideal of A mapped isomorphically to an ideal of B, then $K_n(A, I) \to K_n(B, I)$ need not be an isomorphism if $n > 0$. Having fixed I as a non-unital algebra, A. Suslin [23] showed, by building on earlier work of himself and M. Wodzicki [24], that I satisfies excision for *all* such morphisms $A \to B$ if and only if I is *homologically unital*, in Wodzicki's sense that $\text{Tor}_*^{\mathbb{Z} \ltimes I}(\mathbb{Z}, \mathbb{Z}) = 0$ for $* > 0$. Unfortunately, this is not commonly satisfied for rings of algebraic geometry. A recent emerging trend has therefore been to consider instead the problem of 'pro-excision': i.e., When is the map $\text{``}\varprojlim_r\text{''} \, K_n(A, I^r) \to \text{``}\varprojlim_r\text{''} \, K_n(B, I^r)$ an isomorphism? Our first main theorem assures us that it is always an isomorphism for Noetherian $\mathbb{Q}$-algebras.

Prior to this paper, A. Krishna [13] had established theorem 0.1 in the special case that A is reduced and essentially of finite type over a field of characteristic zero, B is the normalisation of A, and B is assumed to be smooth. He applied this special case of the theorem to the study of algebraic cycles on singular varieties, in particular establishing a higher dimensional version of a conjecture of V. Srinivas concerning the Levine–Weibel Chow groups

[15] and relating the Chow group of a singular variety with that of its desingularisation. We refer to [14] for an overview of related work; in forthcoming work we investigate applications of theorem 0.1 in a similar way. Secondly, T. Geisser and L. Hesselholt [7, 8] have established a pro version of the Suslin–Wodzicki condition; it says that the conclusion of theorem 0.1 holds provided that "$\varprojlim$"$_r$ $\mathrm{Tor}_*^{\mathbb{Z} \ltimes I^r}(\mathbb{Z}, \mathbb{Z}) = 0$ for $* > 0$, a condition which, in the commutative, Noetherian world, is satisfied in 'smooth enough' situations. These two results were the inspiration to prove the first main theorem.

Pro-excision for a one-dimensional ring A and its normalisation B may be proved using Geisser–Hesselholt's condition without any characteristic assumption. Applications of pro-excision in this situation were studied by Krishna [12] in characteristic zero and were the focus of the author's papers [18, 17]: in [18] pro-excision was used to study adelic resolutions of K-theory, while [17] presented numerous applications to the K-theory of one-dimensional singular rings and to Geller's conjecture. The author is currently working on similar applications in higher dimension, using the main results of this paper.

Descent problems for K-theory have a long and distinguished history, which we make no attempt to cover exhaustively. However, of particular interest is descent in V. Voevodsky's h-topology. Recall that the h-topology on Var_k, the category of varieties over a fixed base field k of characteristic zero, is the Grothendieck topology generated by the cd-structure in which a distinguished square is an *abstract blow-up square*, i.e., a cartesian square of k-varieties

$$
\begin{array}{ccc}
Y' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
$$

in which $Y \to X$ is a closed embedding, $X' \to X$ is proper, and $X' \setminus Y' \to X \setminus Y$ is an isomorphism. A presheaf of spectra, or of complexes, $\mathcal{E}$ on Var_k satisfies descent in the h-topology if and only if it carries each such abstract blow-up square to a homotopy cartesian square

$$
\begin{array}{ccc}
\mathcal{E}(X) & \longrightarrow & \mathcal{E}(X') \\
\downarrow & & \downarrow \\
\mathcal{E}(Y) & \longrightarrow & \mathcal{E}(Y')
\end{array}
$$

(see [27]). For example, it was proved by C. Haesemeyer [9] that Weibel's homotopy invariant K-theory satisfies h-descent. His method of proof was subsequently axiomatised in [4, Thm. 3.12] to show that a presheaf of spectra satisfies h-descent as soon as it satisfies excision, is invariant under nilpotent extension, and takes blow-up squares along regularly embedded centres to homotopy cartesian squares. If this axiomatisation could be directly applied in our setting of pro spectra it would allow us to immediately deduce theorem 0.2 from theorem 0.1; unfortunately, technical obstacles prevent it from applying and so theorem 0.2 is proved without appealing to theorem 0.1.

Applications and Examples

In this continuation of the introduction we collect together various applications and special cases of the two main theorems. We begin by summarising the main ways of using pro-excision; analogous statements apply to pro-descent, and to Hochschild/cyclic homology:

Lemma 0.4. *Let $f : A \to B$ be a homomorphism of rings and I an ideal of A mapped isomorphically to an ideal of B. Assume that "$\varprojlim$"$_r K_n(A, I^r) \to$ "$\varprojlim$"$_r K_n(B, I^r)$ is an isomorphism for all $n \in \mathbb{Z}$.*

(i) There is a natural, long exact, Mayer–Vietoris sequence

$$\cdots \to K_n(A) \to \text{``}\varprojlim_r\text{''} K_n(A/I^r) \oplus K_n(B) \to \text{``}\varprojlim_r\text{''} K_n(B/I^r) \to \cdots$$

(ii) Suppose that J (resp. J') is an ideal of A (resp. of B) containing I (resp. $f(I)$). Then there is a natural, long exact, Mayer–Vietoris sequence of relative K-groups

$$\cdots \to K_n(A, J) \to \text{``}\varprojlim_r\text{''} K_n(A/I^r, J/I^r) \oplus K_n(B, J') \to \text{``}\varprojlim_r\text{''} K_n(B/I^r, J'/I^r) \to \cdots$$

(iii) Suppose $J \supseteq I$ is an ideal of A mapped isomorphically to an ideal of B. For any $n \geq 0$, the natural map

$$K_n(A, B, J) \to \text{``}\varprojlim_r\text{''} K_n(A/I^r, B/I^r, J/I^r)$$

is an isomorphism.

(iv) Suppose $J \supseteq I$ is an ideal of A mapped isomorphically to an ideal of B. For any $n \geq 0$, the map $K_n(A, B, J) \to K_n(A/I^r, B/I^r, J/I^r)$ is split injective for $r \gg 0$.

(v) The following square of spectra

$$
\begin{array}{ccc}
K_n(A) & \longrightarrow & K_n(B) \\
\downarrow & & \downarrow \\
\operatorname{holim}_r K_n(A/I^r) & \longrightarrow & \operatorname{holim}_r K_n(B/I^r)
\end{array}
$$

is homotopy cartesian, and thus there are analogues of (i)–(v) in which the formal pro groups are replaced by homotopy groups of homotopy limits.

Proof. These are all deduced easily from the vanishing of the groups "$\varprojlim$"$_r K_n(A, B, I^r)$; the details may be found in [17]. $\qquad\square$

Example 0.5. Let A be a reduced, excellent $\mathbb{Q}$-algebra; let $B = \widetilde{A}$ be its normalisation, and let $I = \operatorname{Ann}_A(B/A)$ be the conductor ideal. Then theorem 0.1 applies, yielding a long exact, Mayer–Vietoris sequence

$$\cdots \to K_n(A) \to \text{``}\varprojlim_r\text{''} K_n(A/I^r) \oplus K_n(B) \to \text{``}\varprojlim_r\text{''} K_n(B/I^r) \to \cdots$$

In other words, *conductor squares give Mayer–Vietoris sequences in K-theory, as long as one allows all infinitesimal thickenings of the conductor.*

A major theme of [17] was understanding when such sequences break into short exact sequences in dimension one. For example, if A is one-dimensional and semi-local, the analogous sequence for the relative groups was shown to break into short exact sequences:

$$0 \to K_n(A, \mathfrak{m}) \to \text{``}\varprojlim_r\text{''} K_n(A/\mathfrak{m}^r, \mathfrak{m}/\mathfrak{m}^r) \oplus K_n(\widetilde{A}, \mathfrak{M}) \to \text{``}\varprojlim_r\text{''} K_n(\widetilde{A}/\mathfrak{M}^r, \mathfrak{M}/\mathfrak{M}^r) \to 0$$

(where $\mathfrak{m}, \mathfrak{M}$ are the Jacobson radicals of A and $\widetilde{A}$ respectively), the consequences of which were given [loc. cit.].

Example 0.6. Let A be a Noetherian $\mathbb{Q}$-algebra containing ideals I, J which have zero intersection; put $M = I + J$. We will use the standard notation

$$K(A; I, J) = K_n(A, A/I, J) \simeq K_n(A, A/J, I).$$

Given $s \geq r \geq 1$, consider the composition

$$K_n(A; I^s, J^s) \to K_n(A; I^s, J^r) \to K_n(A; I^r, J^r).$$

Theorem 0.1 implies that for any n, r, there is $s \geq r$ such that the second arrow, hence the composition, is zero; therefore "$\varprojlim_r$" $K_n(A; I^r, J^r) = 0$ for all n. It follows that

$$\text{"}\varprojlim_r\text{"} K_n(A, I^r) \to \text{"}\varprojlim_r\text{"} K_n(A/J^r, M^r/J^r)$$

is an isomorphism for all n, and hence

$$
\begin{array}{ccc}
K(A) & \longrightarrow & \text{"}\varprojlim_r\text{"} K(A/J^r) \\
\downarrow & & \downarrow \\
\text{"}\varprojlim_r\text{"} K(A/I^r) & \longrightarrow & \text{"}\varprojlim_r\text{"} K(A/M^r)
\end{array}
$$

is homotopy cartesian; i.e., there is a long exact, Mayer–Vietoris sequence

$$\cdots \to K_n(A) \to \text{"}\varprojlim_r\text{"} K(A/J^r) \oplus \text{"}\varprojlim_r\text{"} K(A/I^r) \to \text{"}\varprojlim_r\text{"} K(A/M^r) \to \cdots$$

Example 0.7. A particular case in which the previous example applies is the coordinate axes $A = k[X, Y]/\langle XY \rangle$, where k is any Noetherian $\mathbb{Q}$-algebra. Letting $\widetilde{K}$ denote reduced K-theory, i.e. the quotient by $K(k)$, we have proved that the square

$$
\begin{array}{ccc}
\widetilde{K}(k[X,Y]/\langle XY \rangle) & \longrightarrow & \text{"}\varprojlim_r\text{"} \widetilde{K}(k[X,Y]/\langle XY, Y^r \rangle) \\
\downarrow & & \downarrow \\
\text{"}\varprojlim_r\text{"} \widetilde{K}(k[X,Y]/\langle XY, X^r \rangle) & \longrightarrow & \text{"}\varprojlim_r\text{"} \widetilde{K}(k[X,Y]/\langle X^r, Y^r \rangle)
\end{array}
$$

is homotopy cartesian, and thus there is a long exact, Mayer–Vietoris sequence of the reduced K-groups.

Example 0.8. Let k be a field of characteristic zero, and let X be a separated, integral k-variety. Let $X' \to X$ be a resolution of singularities, and let $Y \subseteq X$ be the reduced closed subscheme outside of which X' is isomorphic to X; put $Y' = X' \times_X Y$. Then theorem 0.2 implies that

$$
\begin{array}{ccc}
K(X) & \longrightarrow & K(X') \\
\downarrow & & \downarrow \\
\text{"}\varprojlim_r\text{"} K(rY) & \longrightarrow & \text{"}\varprojlim_r\text{"} K(rY')
\end{array}
$$

is homotopy cartesian. In particular, we deduce that the homotopy fibre $K(X, X')$ of $K(X) \to K(X')$ is supported on Y in the following strong sense: For each $n \in \mathbb{Z}$, the group $K_n(X, X')$ is a direct summand of $K_n(rY, rY')$ for some $r \geq 0$.

When X is the spectrum of a local ring, stronger results become available: the kernel of $K_n(X, Y) \to K_n(X', Y')$ can be shown to embed into $K_n(rY, Y)$ for $r \gg 0$, in certain cases, offering a higher dimensional generalisation of the author's results described in example 0.5. Such phenomena are to be studied in detail in a later paper.

Method of proof

The proofs of the two main theorems are rather conceptual once the approach is understood.

Theorem 0.1 is equivalent to the statement that the pro birelative groups "$\varprojlim$"$_r K_n(A, B, I^r)$ are zero for all $n \geq 0$. Thanks to G. Cortiñas' proof of the KABI conjecture [3], these birelative groups are isomorphic to $HC_n^{\mathbb{Q}}(A, B, I^r)$, and so theorem 0.1 is immediately reduced to proving the same claim for cyclic homology. In turn, the SBI sequence reduces the problem to Hochschild homology, at which point the Quillen/Hodge/λ-decomposition reduces the problem to proving theorem 0.1 for André–Quillen homology D_*^i in place of K-theory. These reductions are explained in more detail in section 4.

To prove pro-excision for André–Quillen homology we improve some arguments originally due to Krisha [13]. Let $k \to A \to B$ be homomorphisms of Noetherian $\mathbb{Q}$-algebras such that $A \to B$ is of finite type, and let I be an ideal of A mapped isomorphically to an ideal of B. It is required to show that, for each $i \geq 0$, the square of pro cotangent complexes

$$
\begin{array}{ccc}
\mathbb{L}_{A|k}^i & \longrightarrow & \mathbb{L}_{B|k}^i \\
\downarrow & & \downarrow \\
\mathbb{L}_{A/I^\infty|k}^i & \longrightarrow & \mathbb{L}_{B/I^\infty|k}^i
\end{array}
$$

is homotopy cartesian, where we write $\mathbb{L}_{A/I^\infty|k}^i = $ "$\varprojlim$" $\mathbb{L}_{A/I^r|k}^i$ (we will use obvious modifications of this notational device throughout this outline of the proof).

The first step is to establish that $\mathbb{L}_{A/I^\infty|k}^i \simeq \mathbb{L}_{A|k} \otimes_A A/I^\infty$; i.e., that

$$
\text{``}\varprojlim_r\text{''} \, D_*^i(A/I^r|k) \cong \text{``}\varprojlim_r\text{''} \, D_*^i(A|k, A/I^r) \tag{1}
$$

(and similarly for B). Once this is achieved, the bottom of the square of cotangent complexes may be replaced by the map $\mathbb{L}_{A|k}^i \otimes_A A/I^\infty \to \mathbb{L}_{B|k}^i \otimes_B B/I^\infty$ and so, taking homotopy fibres in the vertical direction, it is enough to prove that $\mathbb{L}_{A|k}^i \otimes_A I^\infty \simeq \mathbb{L}_{B|k}^i \otimes_B I^\infty$. This follows from the vanishing result

$$
\mathbb{L}_{B|A}^i \otimes_B I^\infty \simeq 0 \tag{2}
$$

and the Jacobi–Zariski style implication

$$
\mathbb{L}_{B/A}^i \otimes_B I^\infty \simeq 0 \Longrightarrow \mathbb{L}_{A|k}^i \otimes_A I^\infty \simeq \mathbb{L}_{B|k}^i \otimes_B I^\infty \tag{3}
$$

Results of flavour (1) – (3) are the subject of section 1, where theorem 1.7 offers indispensable tools for manipulating formal projective limits of André–Quillen homology groups ("Artin–Rees type properties"). This section also begins with an introduction to André–Quillen homology and outlines are reasons for using pro groups. Of particular note is the 'higher Jacobi–Zariski spectral sequence' proposition 1.12, needed to prove implications like (3); it seems to be originally due to C. Kassel and A. Sletsjøe [11], but we inadvertently reproved it.

With the Artin–Rees properties at hand, the programme just outlined for proving pro-excision for André–Quillen homology is easily achieved in section 5.3. (These Artin–Rees properties also quickly prove theorem 0.3, but we postpone the proof until an appendix to

avoid disrupting the proof of theorem 0.1.) Section 3 then introduces the derived version of Hoschchild homology we use and observes that our Artin–Rees properties remain valid for Hochschild homology. As explained above, 4 then explains in detail how to pass from André–Quillen homology to K-theory.

The proof of theorem 0.2 occupies sections 5–6 and follows a very similar outline to that of theorem 0.1 once André–Quillen homology has been defined for schemes, which is done in the usual way using hypercohomology.

The appendix, as well as proving the pro HKR theorem, introduces the notion of a 'virtually geometrically regular' morphism; this is a natural condition which forces André–Quillen homology groups, hence Hochschild homology groups, to be I-adically Artin–Rees; this in turn implies that long exact Mayer–Vietoris sequences break into short exact sequences, yielding better information about $\mathrm{Ker}(HH_n^k(A) \to HH_n^k(B))$.

The majority of the proofs of both theorems are presented in some generality, treating the case of arbitrary coefficient modules and keeping track of which integer needs to be invertible in order for the results to hold.

Notation, etc.

All rings are commutative and unital, though in many places this is not necessary. The K-groups of a scheme are in the style of R. Thomason and T. Trobaugh [26].

Acknowledgments

1   Artin-Rees properties for André-Quillen homology

The aim of this section is theorem 1.7 below, which collects together the main Artin–Rees type properties of André–Quillen homology, phrased in the language of pro groups. The $i = 1$ case of the theorem follows quickly from the original work of M. André and D. Quillen, while our proof of the $i > 1$ case is based on proofs of similar assertions by A. Krishna [13]; to be precise, he implicitly proved parts of corollary 1.8 under more restrictive assumptions, and section 1.1 uses many ideas found in his work.

We begin with a review of André-Quillen homology [1, 21, 22], though we assume the reader is familiar with its basic properties; let $k \to A$ be a morphism of (commutative, unital) rings. Let $P_\bullet \to A$ be a simplicial resolution of A by free k-algebras, and set

$$\mathbb{L}_{A|k} := \Omega^1_{P_\bullet|k} \otimes_{P_\bullet} A.$$

Thus $\mathbb{L}_{A|k}$ is a simplicial A-module which is free in each degree; it is called the *cotangent complex* of the k-algebra A. The cotangent complex is well-defined up to homotopy, since the free simplicial resolution $P_\bullet \to A$ is unique up to homotopy.

Given simplicial A-modules $M_\bullet$, $N_\bullet$, the tensor product and alternating powers are new simplicial A-modules defined termwise: $(M_\bullet \otimes_A N_\bullet)_n = M_n \otimes_A N_n$ and $(\bigwedge_A^r M_\bullet)_n = \bigwedge_A^r M_n$.

In particular we set $\mathbb{L}_{A|k}^i := \bigwedge_A^i \mathbb{L}_{A|k}$ for each $i \geq 1$. The *André-Quillen homology* of the k-algebra A, with coefficients in any A-module M, is defined by

$$D_n^i(A|k, M) := H_n(\mathbb{L}_{A|k}^i \otimes_A M),$$

for $n \geq 0$, $i \geq 1$. When $M = A$ the notation is simplified to

$$D_n^i(A|k) := D_n^i(A|k, A) = H_n(\mathbb{L}_{A|k}^i).$$

When $i = 1$ the superscript is often omitted, writing $D_n(A|k, M) = H_n(\mathbb{L}_{A|k} \otimes_A M)$ and $D_n(A|k) = H_n(\mathbb{L}_{A|k})$ instead. If $k \to A$ is essentially of finite type and k is Noetherian, then $D_n^i(A|k, M)$ is a finitely generated A-module for all n, i and for all finitely generated A-modules M.

Let $k \to A \to B$ be ring homomorphisms. Then there is an exact sequence of simplicial B-modules

$$0 \to \mathbb{L}_{A/k} \otimes_A B \to \mathbb{L}_{B/k} \to \mathbb{L}_{B/A} \to 0.$$

This remains exact upon tensoring by any B-module M since these simipcial B-modules are free in all degrees; taking homology yields the so-called Jacobi-Zariski long exact sequence

$$\cdots \to D_n(A|k, M|_A) \to D_n(B|k, M) \to D_n(B|A, M) \to \cdots$$

Finally, if $0 \to M \to N \to P \to 0$ is a short exact sequence of A-modules, then there is a resulting long exact sequence for each $i \geq 1$:

$$\cdots \to D_n^i(A|k, M) \to D_n^i(A|k, N) \to D_n^i(A|k, P) \to \cdots$$

Remark 1.1. To avoid any ambiguity once spectral sequences appears, we remark that the notation $D_n^i(A|k, M)$ is defined in the same way if $i \leq 0$ or $n < 0$. However, $D_n^i(A|k, M) = 0$ if $n < 0$ and

$$D_n^0(A|k, M) = \begin{cases} M & n = 0, \\ 0 & \text{else,} \end{cases}$$

since $\mathbb{L}_{A|k}^0 \otimes_A M \simeq M$.

Remark 1.2. A fundamental tool in our exposition is formal pro abelian groups. Everything we need about categories of pro objects may be found in one of the standard references, such as the appendix to [2], or [10]. We will often use $\mathrm{Pro}(A\text{-}mod)$, the category of pro A-modules for some commutative ring A, and $\mathrm{Pro}\,Ab$, the category of pro (abelian) groups. We find this to be a convenient and conceptual way to state many of the results.

If $\mathcal{C}$ is a category, then $\mathrm{Pro}\,\mathcal{C}$, the *category of pro objects of* $\mathcal{C}$, is the following: an object of $\mathrm{Pro}\,\mathcal{C}$ is a contravariant functor $X : \mathcal{I} \to \mathcal{C}$, where $\mathcal{I}$ is a small cofiltered category (it is fine to assume that $\mathcal{I}$ is a codirected set, and in fact all our pro systems will be indexed by the postive integers); this object is usually denoted

$$\text{``}\varprojlim_{i \in \mathcal{I}}\text{''}\, X(i) \qquad \text{or} \qquad \text{``}\varprojlim_i\text{''}\, X(i),$$

or by some other suggestive notation[1]. The morphisms in $\mathrm{Pro}\,\mathcal{C}$ from $\text{``}\varprojlim\text{''}_{i \in \mathcal{I}} X$ to $\text{``}\varprojlim\text{''}_{j \in \mathcal{J}} Y(j)$ are

$$\mathrm{Hom}_{\mathrm{Pro}\,\mathcal{C}}(\text{``}\varprojlim_{i \in \mathcal{I}}\text{''}\, X, \text{``}\varprojlim_{j \in \mathcal{J}}\text{''}\, Y) := \varprojlim_{j \in \mathcal{J}} \varinjlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(X(i), Y(j)),$$

---

[1]The notation $\{X(i)\}_i$ seems to be favoured in the homotopy community.

where the right side is a genuine pro-ind limit in the category of sets. Composition is defined in the obvious way.

There is a fully faithful embedding $\mathcal{C} \to \operatorname{Pro}\mathcal{C}$. Assuming that projective limits exist in $\mathcal{C}$, there is a realisation functor

$$\operatorname{Pro}\mathcal{C} \to \mathcal{C}, \quad \text{``}\varprojlim_{i \in \mathcal{I}}\text{''}\, X(i) \mapsto \varprojlim_{i \in \mathcal{I}} X(i),$$

which is left exact but not right exact (its derived functors are precisely $\varprojlim^1, \varprojlim^2$, etc.), and which is a left adjoint to the aforementioned embedding.

Suppose that $\mathcal{A}$ is an abelian category. Then $\operatorname{Pro}\mathcal{A}$ is an abelian category. Moreover, given a system of exact sequences

$$\cdots \longrightarrow X_{n-1}(i) \longrightarrow X_n(i) \longrightarrow X_{n+1}(i) \longrightarrow \cdots,$$

the formal limit

$$\cdots \longrightarrow \text{``}\varprojlim_{i \in \mathcal{I}}\text{''}\, X_{n-1}(i) \longrightarrow \text{``}\varprojlim_{i \in \mathcal{I}}\text{''}\, X_n(i) \longrightarrow \text{``}\varprojlim_{i \in \mathcal{I}}\text{''}\, X_{n+1}(i) \longrightarrow \cdots$$

is an exact sequence in $\operatorname{Pro}\mathcal{A}$. Of course, we cannot deduce that

$$\cdots \longrightarrow \varprojlim_{i \in \mathcal{I}} X_{n-1}(i) \longrightarrow \varprojlim_{i \in \mathcal{I}} X_n(i) \longrightarrow \varprojlim_{i \in \mathcal{I}} X_{n+1}(i) \longrightarrow \cdots$$

is exact in $\mathcal{A}$ (assuming that all these projective limits exist in $\mathcal{A}$) because the realisation functor is not right exact.

Example 1.3. Suppose that $A = \text{``}\varprojlim_r\text{''}\, A(r)$ is a pro group indexed by the positive integers $r \geq 1$. Then $A \cong 0$ in $\operatorname{Pro}Ab$ if and only if for each $r \geq 1$ there exists $s \geq r$ for which the transition map $A(s) \to A(r)$ is zero.

Example 1.4. As an illustration of the use of pro groups, we consider a spectral sequence example. Suppose that

$$E^1_{pq}(r) \Longrightarrow H_{p+q}(r),$$

for $r \geq 1$, are spectral sequences in some abelian category $\mathcal{A}$, which are functorial in that we have morphisms of spectral sequences $E^\bullet_{pq}(1) \leftarrow E^\bullet_{pq}(2) \leftarrow \cdots$. To avoid convergence issues, suppose that each spectral sequence is bounded, by a bound independent of r; e.g., each spectral sequence might be zero outside the first quadrant. Finally, make the following assumption:

> For each p, q and each $r \geq 1$, there exists $s \geq r$ such that $E^1_{pq}(s) \to E^1_{pq}(r)$ is zero.

Then we claim that for each n and each $r \geq 1$, there exists $s \geq r$ such that $H_n(s) \to H_n(r)$ is zero. We offer two proofs of this claim:

Careful proof: For simplicity of notation, assume that each spectral sequence is zero outside the first quadrant, and let

$$H_n(r) = F_n H_n(r) \supseteq \cdots \supseteq F_{-1} H_n(r) = 0$$

denote the resulting filtration on each $H_n(r)$. By the natural dependence of the spectral sequences on r, the natural maps $H_n(s) \to H_n(r)$, for $s \geq r$, respect the filtrations and thus induce homomorphisms $\mathrm{gr}_p H_n(s) \to \mathrm{gr}_p H_n(r)$ for $p = 0, \ldots, n$. But $\mathrm{gr}_p H_n(r)$ is a subquotient of $E^1_{pq}(r)$, and same for s, so our standing assumption implies that we may pick $s \geq r$ such that $\mathrm{gr}_p H_n(s) \to \mathrm{gr}_p H_n(r)$ is zero.

So, starting with any $r \geq 1$, successively pick integers $r_n \geq \cdots \geq r_0 = r$ such that $\mathrm{gr}_{n-i} H_n(r_i) \to \mathrm{gr}_{n-i} H_n(r_{i-1})$ is zero for $i = 1, \ldots, n$. In other words, the natural map $H_n(r_i) \to H_n(r_{i-1})$ carries $F_{n-i}H_n(r_i)$ to $F_{n-i+1}H_n(r_{i-1})$. Hence the composite

$$H_n(r_n) \to H_n(r_{n-1}) \to \cdots \to H_n(r_1) \to H_n(r)$$

carries $F_n H_n(r_n) = H_n(r_n)$ to $F_{-1}H_n(r) = 0$. So setting $s = r_n$ completes the proof of the claim.

Slick proof: The naturality of the family of spectral sequences and the exactness of "$\varprojlim$"$_r$ implies that we obtain a first quadrant spectral sequence

$$\text{``}\varprojlim_r\text{''} \, E^1_{pq}(r) \Longrightarrow \text{``}\varprojlim_r\text{''} \, H_{p+q}(r)$$

in Pro Ab. But our assumption, rephrased into the pro language, is exactly that "$\varprojlim$"$_r \, E^1_{pq}(r) = 0$ for each p, q. Hence "$\varprojlim$"$_r \, H_n(r) = 0$ for each n, which is exactly the claim.

In our subsequent calculations we will systemically make use of pro arguments of this type, usually without explicit mention; moreover, we will state our results in terms of pro objects, rather than "for each $r \geq 1$ there exists $s \geq r$ such that etc.", whenever it does not cause confusion.

Now we turn to Artin-Rees type properties, starting by recalling the following definition:

Definition 1.5. (M. André [1, §X]) Let I be an ideal of a ring A and let M be an A-module. We will say that M is *Artin-Rees for the I-adic topology*, or more simply *I-adically Artin-Rees*, if and only if whenever $n \geq 1$ and $r \geq 1$ are given, there exists $s \geq r$ such that the natural map

$$\mathrm{Tor}^A_n(M, A/I^s) \longrightarrow \mathrm{Tor}^A_n(M, A/I^r)$$

is zero. In other words, M is I-adically Artin-Rees if and only if "$\varprojlim$"$_r \, \mathrm{Tor}_n(M, A/I^r) = 0$ for all $n \geq 1$.

Remark 1.6. If A is Noetherian then any finitely generated A-module is I-adically Artin–Rees [1, Lem. X.11] (proved by appealing to the usual Artin–Rees property of each term in a resolution of the module by finitely generated, projective A-modules).

Our first goal is to prove the following Artin–Rees type properties for André–Quillen homology, where we adopt the notation $D^i_n(-)' := D^i_n(-) \otimes_{\mathbb{Z}} \mathbb{Z}[1/i!]$. Of course, if $i!$ is invertible in the ring under consideration then $D^i_n(-)' = D^i_n(-)$ (and this equality always holds when $i = 1$).

Theorem 1.7. *Let $k \to A$ be a ring homomorphism, let I be an ideal of A, let M be an A-module, and fix an integer $i \geq 1$. Consider the following statements:*

(i) Let W be an A/I-module. Then "$\varprojlim$"$_r \, D^i_n(A/I^r|A, W)' = 0$ for all $n \geq 0$.

(ii) *The natural map*

$$\text{``}\varprojlim_{r}\text{''}\, D_n^i(A|k, M/I^r M)' \longrightarrow \text{``}\varprojlim_{r}\text{''}\, D_n^i(A/I^r|k, M/I^r M)'$$

is an isomorphism for all $n \geq 0$.

(iii) *The natural map*

$$\text{``}\varprojlim_{r}\text{''}\, D_n^i(A|k, M) \otimes_A A/I^r \longrightarrow \text{``}\varprojlim_{r}\text{''}\, D_n^i(A|k, M/I^r M)$$

is an isomorphism for all $n \geq 0$.

(i) and (ii) are true if A/I is I-adically Artin-Rees (e.g., A Noetherian suffices by remark 1.6). (iii) is true if $D_n^i(A|k, M)$ is I-adically Artin-Rees for all $n \geq 0$ (e.g., it is enough for M to be finitely generated over A and $k \to A$ to be essentially of finite type (or to be v.g.r., see appendix A.1)).

In the special case $A = M$ we get:

Corollary 1.8. *Let $k \to A$ be a ring homomorphism, let I be an ideal of A, and fix an integer $i \geq 1$. Consider the following statements:*

(i) *$\text{``}\varprojlim_{r}\text{''}\, D_n^i(A/I^r|A, M/I^r M)' = 0$ for all $n \geq 0$ and all A-modules M. In particular,*

$$\text{``}\varprojlim_{r}\text{''}\, D_n^i(A/I^r|A)' = 0$$

for all $n \geq 0$.

(ii) *The natural map*

$$\text{``}\varprojlim_{r}\text{''}\, D_n^i(A|k, A/I^r A)' \longrightarrow \text{``}\varprojlim_{r}\text{''}\, D_n^i(A/I^r|k)'$$

is an isomorphism for all $n \geq 0$.

(iii) *The natural map*

$$\text{``}\varprojlim_{r}\text{''}\, D_n^i(A|k) \otimes_A A/I^r \longrightarrow \text{``}\varprojlim_{r}\text{''}\, D_n^i(A|k, A/I^r A)$$

is an isomorphism for all $n \geq 0$.

(i) and (ii) are true if A/I is I-adically Artin-Rees. (iii) is true if $D_n^i(A|k)$ is I-adically Artin-Rees for all $n \geq 0$.

Proof. To prove (i), pick $r \geq 1$ and apply part (i) of the theorem to the ideal I^r and A/I^r-module $W = M/I^r M$. One deduces that there is $s \geq 1$ such that $D_n^i(A/I^{sr}|k, M/I^r M) \to D_n^i(A/I^r|k, M/I^r M)$ is zero; hence

$$D_n^i(A/I^{sr}|A, M/I^{sr}M) \to D_n^i(A/I^{sr}|A, M/I^r M) \to D_n^i(A/I^r|A, M/I^r M)$$

is zero, which proves the first claim in (i). For the 'in particular' claim, simply take $M = A$.
Parts (ii) and (iii) are also proved by taking $M = A$. □

Remark 1.9. For the sake of completeness, notice that parts (ii) and (iii) of the theorem and corollary are true when $i = 0$, regardless of any assumptions.

Remark 1.10. For the reader less familiar with formal pro groups, we offer a down-to-earth example. Suppose that (ii) and (iii) of the previous corollary hold, and assume that $i!$ is invertible in A for simplicity. Then the natural map

$$\text{``}\varprojlim_r\text{''}\, D_n^i(A|k) \otimes_A A/I^r \to \text{``}\varprojlim_r\text{''}\, D_n^i(A/I^r|k)$$

is an isomorphism for all $n \geq 0$. The surjectivity of this map means that the formal projective limit of the groups

$$D_n^i(A/I^r|k)/\text{Im. of } D_n^i(A|k)$$

is zero. In other words, for each r there exists $s \geq r$ with the following property: If an element of $D_n^i(A/I^r|k)$ can be lifted to $D_n^i(A/I^s|k)$ then it can actually be lifted to $D_n^i(A|k)$. Similarly, the injectivity of the map means that for each r there exists $s \geq r$ with the following property: if an element of $D_n^i(A|k)$ vanishes in $D_n^i(A/I^s|k)$, then it belongs to $I^r D_n^i(A|k)$.

Remark 1.11. The previous theorem and corollary may be stated directly in terms of the pro cotangent complexes. Given a projective system $X_\bullet(1) \leftarrow X_\bullet(2) \leftarrow \cdots$ of simplicial A-modules, we may take the limit in each degree to form $\text{``}\varprojlim_r\text{''}\, X_\bullet(r)$, which is a simplicial object in the abelian category of pro A-modules. By construction, its homology is

$$H_n(\text{``}\varprojlim_r\text{''}\, X_\bullet(r)) = \text{``}\varprojlim_r\text{''}\, H_n(X_\bullet(r)).$$

From the notation, the object $\text{``}\varprojlim_r\text{''}\, X_\bullet(r)$ appears to live in the abelian category $\mathrm{Pro}(A\text{-}mod^{\Delta^{\mathrm{op}}})$ (pro objects in the category of simplicial A-modules), but we prefer to view it in $\mathrm{Pro}(A\text{-}mod)^{\Delta^{\mathrm{op}}}$ (simplicial objects in the category of pro A-modules) via the natural functor

$$\mathrm{Pro}(A\text{-}mod^{\Delta^{\mathrm{op}}}) \to \mathrm{Pro}(A\text{-}mod)^{\Delta^{\mathrm{op}}}.$$

We do this because terms like 'acyclic' are already defined in $\mathrm{Pro}(A\text{-}mod)^{\Delta^{\mathrm{op}}}$; otherwise we would have to introduce a suitable model structure on $\mathrm{Pro}(A\text{-}mod^{\Delta^{\mathrm{op}}})$. (Actually, these two categories are likely to be the same, but our convention means that such an issue is irrelevant.)

The first two statements of the previous theorem are equivalent to the following (assuming $i!$ is invertible in A to avoid writing extra localisations):

(i) $\text{``}\varprojlim_r\text{''}\, \left(\mathbb{L}_{A/I^r|A}^i \otimes_{A/I^r} W\right)$ is acyclic.

(ii) $\text{``}\varprojlim_r\text{''}\, \left(\mathbb{L}_{A|k}^i \otimes_A M/I^r M\right) \simeq \text{``}\varprojlim_r\text{''}\, \left(\mathbb{L}_{A/I^r|k}^i \otimes_{A/I^r} M/I^r M\right)$

We will take advantage of this pro simplicial framework when we turn to excision in section 5.3; it is not essential but vastly simplifies the exposition.

1.1 Proof of theorem 1.7

This section is dedicated to the proof of theorem 1.7, following Krishna's methods in [13]. We begin by treating part (iii), which requires the Universal Coefficient spectral sequence:

Universal Coefficient spectral sequence: Let $k \to A$ be a morphism of rings, and let M, N be A-modules. Then there is a natural, first quadrant spectral sequence

$$E^2_{pq} = \operatorname{Tor}^A_p(D^i_q(A|k, M), N) \implies D^i_{p+q}(A|k, M \otimes_A N).$$

(We call this the Universal Coefficient spectral sequence because of the special case $M = A$.) It is proved in the usual way by choosing a simplicial resolution of N by flat A-algebras and applying the 'tensor product spectral sequence', which will be mentioned below.

Proof of part (iii) of the theorem. By hypothesis, we have a morphism $k \to A$, an A-module M, an ideal $I \subseteq A$, and an integer $i \geq 1$ such that $D^i_n(A|k, M)$ is I-adically Artin-Rees for all $n \geq 0$. The Universal Coefficient spectral sequences for the A-modules $M/I^r M = M \otimes_A A/I^r$ take the form

$$E^2_{pq}(r) = \operatorname{Tor}^A_p(D^i_q(A|k, M), A/I^r) \implies D^i_{p+q}(A|k, M/I^r M)$$

Taking the formal limit over r yields a spectral sequence in Pro Ab:

$$\text{``}\varprojlim_r\text{''} \operatorname{Tor}^A_p(D^i_q(A|k, M), A/I^r) \implies \text{``}\varprojlim_r\text{''} D^i_{p+q}(A|k, M/I^r M)$$

(there is no convergence problem since all the spectral sequences are restricted to the first quadrant). By the Artin-Rees hypothesis this pro spectral sequence is zero except along the $p = 0$ column. In other words, the limit of the edge maps

$$\text{``}\varprojlim_r\text{''} D^i_n(A|k, M) \otimes_A A/I^r \to \text{``}\varprojlim_r\text{''} D^i_n(A|k, M/I^r M)$$

is an isomorphism for any $n \geq 0$. □

When $i = 1$, parts (i) and (ii) of the theorem follow immediately from the original work of André and Quillen:

Proof of parts (i) & (ii) of the theorem when $i = 1$. Let I be an ideal of a ring A and assume that A/I is I-adically Artin-Rees. Let W be an A/I-module. Then [1, Prop. 12 – Thm. 14] states that for any $r \geq 1$ there exists $s \geq r$ such that the natural map

$$D_n(A/I^s|A, W) \longrightarrow D_n(A/I^r|A, W)$$

is zero. In other words, "$\varprojlim_r$" $D_n(A/I^r|A, W) = 0$, proving part (i).

Next suppose that A is a k-algebra for some ring k, and let M be an A-module. Then the long exact Jacobi–Zariski sequence for $k \to A \to A/I^r$ is

$$\cdots \to D_n(A|k, M/I^r M) \to D_n(A/I^r|k, M/I^r M) \to D_n(A/I^r|A, M/I^r M) \to \cdots$$

Corollary 1.8(i) for $i = 1$, which follows from what we have just proved, shows that "$\varprojlim_r$" of the right-most term is zero; so taking the limit over r completes the proof. □

It remains to prove parts (i) and (ii) of the theorem when $i > 1$. This is achieved via a number of spectral sequences, which we will introduce as they are needed:

Tensor product spectral sequence: Suppose that A is a commutative ring and $M_\bullet$, $N_\bullet$ are complexes of A-modules supported in positive degree. Then there is a natural, first quadrant spectral sequence

$$E_{pq}^0 = N_p \otimes_A M_q \Longrightarrow H_{p+q}(N_\bullet \otimes_A M_\bullet),$$

whose first page is $E_{pq}^1 = H_q(N_p \otimes_A M_\bullet)$ (it is the usual spectral sequence of a bicomplex). In particular if each N_p, $p \geq 0$, is flat over A, we obtain the spectral sequence

$$E_{pq}^1 = N_p \otimes_A H_q(M_\bullet) \Longrightarrow H_{p+q}(N_\bullet \otimes_A M_\bullet).$$

This applies equally well if $M_\bullet$, $N_\bullet$ are simplicial A-modules.

This is sufficient to prove part (i) of the theorem in full generality:

Proof of part (i) of the theorem. Let I be an ideal of a ring A and assume that A/I is I-adically Artin-Rees. Let W be an A/I-module. We will first prove by induction on $i \geq 1$ that "$\varprojlim$"$_r H_n(\mathbb{L}_{A/I^r|A}^{\otimes i} \otimes_A W) = 0$ for all $n \geq 0$.

The base case $i = 1$ is just the $i = 1$ case of the theorem, which have already proved. Suppose the result has been proved for some $i > 1$. Since $\mathbb{L}_{A/I^r|A}$ is a simplicial A-module which is free in each degree, the previous remark furnishes us with a first quadrant spectral sequence

$$E_{pq}^1(r) = \mathbb{L}_{A/I^r|A,p} \otimes_{A/I^r} H_q(\mathbb{L}_{A/I^r|A}^{\otimes i-1} \otimes_A W) \Longrightarrow H_{p+q}(\mathbb{L}_{A/I^r|A}^{\otimes i} \otimes_A W)$$

Taking the limit over r yields a spectral sequence in ProAb:

$$E_{pq}^1 = \text{``}\varprojlim_r\text{''} \left( \mathbb{L}_{A/I^r|A,p} \otimes_{A/I^r} H_q(\mathbb{L}_{A/I^r|A}^{\otimes i-1} \otimes_A W) \right) \Longrightarrow \text{``}\varprojlim_r\text{''} H_{p+q}(\mathbb{L}_{A/I^r|A}^{\otimes i} \otimes_A W)$$

But the inductive hypothesis implies that for any $q \geq 0$ and any $r \geq 1$ there exists $s \geq r$ such that $H_q(\mathbb{L}_{A/I^s|A}^{\otimes i-1} \otimes_A W) \to H_q(\mathbb{L}_{A/I^r|A}^{\otimes i-1} \otimes_A W)$ is zero, and so $E_{pq}^1 = 0$ for all p, q. This completes the proof of our first claim.

Next remember that if M is a module over a ring R, there is a natural commutative diagram

$$\bigwedge_R^i M \longrightarrow M^{\otimes i}$$
$$\searrow{\scriptstyle \times i!} \qquad \downarrow$$
$$\bigwedge_R^i M$$

where the top arrow is $m_1 \wedge \cdots \wedge m_j \mapsto \sum_{\sigma \in \text{Sym}(i)} m_{\sigma 1} \otimes \cdots \otimes m_{\sigma i}$ and the diagonal arrow is multiplication by $i!$. This diagram is completely functorial, remains valid after tensoring by a second R-module, and may be applied to simplicial R-modules. Hence for any $n \geq 0$ and $s \geq r \geq 1$, the map

$$D_n^i(A/I^s|A, W) \longrightarrow D_n^i(A/I^r|A, W)$$
$$\parallel \qquad\qquad\qquad\qquad\qquad\qquad \parallel$$
$$H_n(\mathbb{L}_{A/I^s|A}^i \otimes_{A/I^s} W) \longrightarrow H_n(\mathbb{L}_{A/I^r|A}^i \otimes_{A/I^r} W) \xrightarrow{\times i!} H_n(\mathbb{L}_{A/I^r|A}^i \otimes_{A/I^r} W)$$

factors through

$$H_n(\mathbb{L}_{A/I^s|A}^{\otimes i} \otimes_{A/I^s} W) \to H_n(\mathbb{L}_{A/I^r|A}^{\otimes i} \otimes_{A/I^r} W),$$

which is the zero map for $s \gg r$ by the proof of our first claim. This completes the proof (and reveals our reason for replacing the higher André-Quillen homology groups by $D_n^i(-) \otimes_{\mathbb{Z}} \mathbb{Z}[1/i!]$). $\qquad\square$

We proved part (ii) of the theorem in the special case $i = 1$ by applying part (i) to the Jacobi-Zariski long exact sequence. The same will work when $i > 1$, but it requires us to first develop a spectral sequence analogue of the Jacobi-Zariski long exact sequence.

Alternating product spectral sequence: Let A be a ring, let M be an A-module, and let $0 \to L'_\bullet \to L_\bullet \to L''_\bullet \to 0$ be a sequence of simplicial A-modules which is split exact in each degree (the splittings need not be compatible in any way; e.g., it suffices that A be a field, or more generally that each L''_n be a free A-module). Then, for each $i \geq 1$, there is a natural (in particular, independent of the splittings), third octant, bounded spectral sequence of A-modules

$$E_{pq}^0 = \left(\bigwedge^{-p} L'_{p+q} \otimes_A \bigwedge^{i+p} L''_{p+q} \right) \otimes_A M \implies H_{p+q}\left(\left(\bigwedge^i L_\bullet \right) \otimes_A M \right),$$

whose E^1 page is

$$E_{pq}^1 = H_{p+q}\left(\left(\bigwedge^{-p} L'_\bullet \otimes_A \bigwedge^{i+p} L''_\bullet \right) \otimes_A M \right).$$

This is proved as follows: Given any short exact sequence $0 \to L' \to L \to L'' \to 0$ of A-modules, we may define a decreasing filtration on $\bigwedge^i L$ as follows: $F^p \bigwedge^i L$ is the submodule of $\bigwedge^i L$ generated by symbols $m_1 \wedge \cdots \wedge m_i$ where at least p of the elements $m_1, \ldots, m_i$ belong to $L' \subseteq L$. There is a natural, well-defined, surjective homomorphism of A-modules

$$\bigwedge^p L' \otimes_A \bigwedge^{i-p} L'' \to \mathrm{gr}^p \bigwedge^i L, \quad m_1 \wedge \cdots \wedge m_p \otimes m_{p+1} \wedge \cdots \wedge m_i \mapsto m_1 \wedge \cdots \wedge m_p \wedge \widetilde{m}_{p+1} \wedge \cdots \wedge \widetilde{m}_i,$$

where $\widetilde{m}$ denotes an arbitrary lift to L of an element $m \in L''$. In the special case in which $0 \to L' \to L \to L'' \to 0$ splits, it is easy to check that this map is moreover injective, hence an isomorphism. So, filtering $\bigwedge^i L_\bullet$ degree-wise in this fashion thus results in a filtered simplicial A-module whose graded pieces are $\mathrm{gr}^p \bigwedge^i L_\bullet \cong \bigwedge^p L'_\bullet \otimes_A \bigwedge^{i-p} L''_\bullet$. The usual spectral sequence for a filtered simplicial module/complex completes the proof in the case $M = A$. For the case of an arbitrary A-module M, just repeat the argument, using the analogous filtration on $(\bigwedge^i L) \otimes_A M$ instead.

Proposition 1.12 (Higher Jacobi-Zariski spectral sequence)**.** *Let $k \to A \to B$ be ring homomorphisms, let M be a B-module, and fix $i \geq 1$. Then there is a natural, third octant, bounded spectral sequence of A-modules*

$$E_{pq}^1 \implies D_{p+q}^i(B|k, M)$$

whose columns may be described as follows:

  (i) *Suppose $p < -i$ or $p > 0$. Then $E_{pq}^1 = 0$.*

  (ii) *Suppose $p = -i$; i.e., we are on the left-most column of the E^1-page. Then $E_{pq}^1 = D_{q-i}^i(A|k, M|_A)$.*

(iii) *Suppose $-i < p \leq 0$. Then the pth column of the E^1-page is given by a first quadrant spectral sequence*

$$\mathcal{E}^2_{\alpha\beta} = D^{-p}_\alpha\big(A|k, D^{i+p}_\beta(B|A,M)\big) \Longrightarrow E^1_{p,\alpha+\beta-p}.$$

Proof. As recalled at the start of the section, there is the natural Jacobi-Zariski exact sequence of simplicial B-modules

$$0 \to \mathbb{L}_{A/k} \otimes_A B \to \mathbb{L}_{B/k} \to \mathbb{L}_{B/A} \to 0.$$

According to our 'alternating product spectral sequence' above, there is therefore a natural, third octant, bounded spectral sequence of A-modules

$$E^1_{pq} = H_{p+q}\left(\mathbb{L}^{-p}_{A|k} \otimes_A \mathbb{L}^{i+p}_{B|A} \otimes_B M\right) \Longrightarrow H_{p+q}(\mathbb{L}^i_{B|k} \otimes_B M) = D^i_{p+q}(B|k, M).$$

We now fix p, q in the third quadrant and investigate E^{pq}_1.

Certainly $E^1_{pq} = 0$ if $p < -i$, proving (i). On the other hand, if $p = -i$, then $\mathbb{L}^{i+p}_{B|A} = \mathbb{L}^0_{B|A} \simeq B$; thus we have a quasi-isomorphism

$$\mathbb{L}^{-p}_{A|k} \otimes_A \mathbb{L}^{i+p}_{B|A} \otimes_B M \simeq \mathbb{L}^{-p}_{A|k} \otimes_A M|_A,$$

and so $E^1_{pq} = D^i_{q-i}(A|k, M|_A)$, proving (ii).

It remains to prove (iii), so suppose that $p > -i$. Since $\mathbb{L}^{-p}_{A|k}$ is a simplicial A-module which is free in each degree, the 'tensor product spectral sequence' above (applied to $\mathbb{L}^{-p}_{A|k}$ and $\mathbb{L}^{i+p}_{B|A} \otimes_B M$) gives us a natural, first quadrant spectral sequence

$$\mathcal{E}^1_{\alpha\beta} = \mathbb{L}^{-p}_{A|k,\alpha} \otimes_A D^{i+p}_\beta(B|A, M) \Longrightarrow H_{\alpha+\beta}\left(\mathbb{L}^{-p}_{A|k} \otimes_A \mathbb{L}^{i+p}_{B|A} \otimes_B M\right) = E^1_{p,\alpha+\beta-p}.$$

The second page is

$$\mathcal{E}^2_{\alpha\beta} = D^{-p}_\alpha\big(A|k, D^{i+p}_\beta(B|A, M)\big) \Longrightarrow E^1_{p,\alpha+\beta-p},$$

as required.   □

Remark 1.13. Suppose that $k \to A \to B$, M, i are as in the previous proposition, and assume that A is filtered inductive limit of smooth, finite-type k-algebras. Then $\mathbb{L}^{-p}_{A|k} \simeq \Omega^{-p}_{A|k}$ for all p by the Hochschild–Kostant–Rosenberg theorem [16, Thm. 3.5.6], which are flat A-modules, so the spectral sequence simplifies to

$$E^1_{pq} = \Omega^{-p}_{A|k} \otimes_A D^{i+p}_{p+q}(B|A, M) \Longrightarrow D^i_{p+q}(B|k, M).$$

If $k \to K$ is moreover faithfully flat, then the proof of theorem A.3 shows that $E^1_{pq} = E^\infty_{pq}$.

In the special case that $k \to A$ is an extension of characteristic zero fields, I learned this simplified spectral sequence from Krishna [13] and modified his proof to obtain the general higher Jacobi–Zariski spectral sequence of the proposition. However, during the preparation of this paper I later discovered that this general spectral sequence was also established by C. Kassel and A. Sletsjøe [11], who also noted its simplification whenever the HKR theorem could be applied to $k \to A$. Remarkably, it seems not to have been widely applied (a notable exception is [5]).

Using the higher Jacobi–Zariski spectral sequence we may complete the proof of theorem 1.7:

Proof of part (ii) of the theorem. Let $k \to A$ be a homomorphism of rings, M an A-module, an $I \subseteq A$ an ideal such that A/I is I-adically Artin-Rees; fix $i \geq 1$. In this proof we will write $G'' = G \otimes_{\mathbb{Z}} \mathbb{Z}[1/i!]$ for any abelian group G.

For each $r \geq 1$ we apply the previous proposition to the ring homomorphisms $k \to A \to A/I^r$ and the A/I^r-modules $M/I^r M$ to obtain natural, third octant, bounded spectral sequences

$$E^1_{pq}(r) \Longrightarrow D^i_{p+q}(A/I^r | k, M/I^r M), \qquad (\dagger)$$

with the properties described by the proposition.

We claim that "$\varprojlim_r$" $E^1_{pq}(r)'' = 0$ if $p > -i$. It is sufficient to consider the situation $-i < p \leq 0$, in which case the previous proposition tell us that there are natural, first quadrant spectral sequences for all $r \geq 1$:

$$\mathcal{E}^2_{\alpha\beta}(r) = D^{-p}_\alpha\big(A|k, D^{i+p}_\beta(A/I^r|A, M/I^r M)\big) \Longrightarrow E^1_{p,\alpha+\beta-p}$$

Inverting $i!$ and passing to the limit yields a spectral sequence in ProAb,

$$\mathcal{E}^2_{\alpha\beta} = \text{``}\varprojlim_r\text{''}\,\mathcal{E}^2_{\alpha\beta}(r)'' \Longrightarrow \text{``}\varprojlim_r\text{''}\,E^1_{p,\alpha+\beta-p}(r)'',$$

and so to prove our vanishing claim it is enough to show that $\mathcal{E}^2_{\alpha\beta} = 0$ for all α, β. But, as explained in corollary 1.8, part (i) of the theorem (which we have already proved) implies that for any r there exists $s \geq r$ such that the map

$$\mathcal{E}^2_{\alpha\beta}(s) = D^{i+p}_\beta(A/I^s|A, M/I^s M) \to D^{i+p}_\beta(A/I^r|A, M/I^r M) = \mathcal{E}^2_{\alpha\beta}(r)$$

is zero after inverting $(i+p)!$. Since $i + p \leq i$ we deduce that $\mathcal{E}^2_{\alpha\beta}(s)'' \to \mathcal{E}^2_{\alpha\beta}(r)''$ is zero; hence $\mathcal{E}^2_{\alpha\beta} = 0$, completing the proof that "$\varprojlim_r$" $E^1_{pq}(r)'' = 0$ for $p > -i$.

We now form the projective limit of the spectral sequences ($\dagger$) after inverting $i!$:

$$E^1_{pq} = \text{``}\varprojlim_r\text{''}\,E^1_{pq}(r)'' \Longrightarrow \text{``}\varprojlim_r\text{''}\,D^i_{p+q}(A/I^r|k, M/I^r M)''$$

By what we have just proved, and from the previous proposition, this spectral sequence is everywhere zero on the first page except along the column $p = -i$, where it equals

$$E^1_{-i,q} = \text{``}\varprojlim_r\text{''}\,E^1_{-i,q}(r)'' = \text{``}\varprojlim_r\text{''}\,D^i_{q-i}(A|k, M/I^r M)''.$$

Therefore the edge map

$$\text{``}\varprojlim_r\text{''}\,D^i_n(A|k, M/I^r M)'' \to \text{``}\varprojlim_r\text{''}\,D^i_n(A/I^r|k, M/I^r M)''$$

is an isomorphism, as required to complete the proof of theorem 1.7(ii). $\square$

2 PRO-EXCISION FOR ANDRÉ-QUILLEN HOMOLOGY

Let $k \to A \to B$ be morphisms of Noetherian rings such that $A \to B$ is essentially of finite type; suppose that I is an ideal of A mapped isomorphically to an ideal of B. Let M be a finitely generated A-module, and write $M_B = M \otimes_A B$. This notation is fixed throughout this section, in which we will establish that André-Quillen homology satisfies pro-excision in such situations (theorem 2.2).

Lemma 2.1. *There are natural isomorphisms*

$$\text{``}\varprojlim_r\text{''} \, D_n^i(B|A, I^r M_B) \xrightarrow{\cong} \text{``}\varprojlim_r\text{''} \, I^r D_n^i(B|A, M_B) \stackrel{(*)}{\cong} 0$$

and

$$\text{``}\varprojlim_r\text{''} \, D_n^i(A|k, I^r M) \xrightarrow{\cong} \text{``}\varprojlim_r\text{''} \, D_n^i(B|k, I^r M_B)$$

for all $n, i \geq 0$ (except that isomorphism $(*)$ is not valid if $i = n = 0$).

Proof. When $i = 0$ the first isomorphism is trivial, and the third one will follow when we establish that $I^r M \cong I^r M_B$ for $r \gg 0$ at the end of the proof; so assume $i > 0$ henceforth.

The short exact sequence

$$0 \to I^r M_B \to M_B \to M_B / I^r M_B \to 0$$

induces a long exact sequence in André-Quillen homology:

$$\cdots \to D_n^i(B|A, I^r M_B) \to D_n^i(B|A, M_B) \xrightarrow{(*)} D_n^i(B|A, M_B / I^r M_B) \to \cdots$$

When we take the limit over r, map $(*)$ becomes surjective thanks to theorem 1.7(iii) (see also remark 1.10) and we obtain short exact sequences

$$0 \to \text{``}\varprojlim_r\text{''} \, D_n^i(B|A, I^r M_B) \to D_n^i(B|A, M_B) \to \text{``}\varprojlim_r\text{''} \, D_n^i(B|A, M_B) \otimes_B B/I^r \to 0.$$

This proves that

$$\text{``}\varprojlim_r\text{''} \, D_n^i(B|A, I^r M_B) \cong \text{``}\varprojlim_r\text{''} \, I^r D_n^i(B|A, M_B).$$

Next we will show that for any given n, the module $I^r D_n^i(B|A, M_B)$ is zero for $r \gg 0$. Since $D = D_n^i(B|A, M_B)$ is a finitely generated B-module, it is sufficient to show that $D_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} B$ not containing I. Let $\mathfrak{p}$ be such a prime ideal of B, and let $\mathfrak{q}$ be its contraction to A. Since André-Quillen homology commutes with localisation of the base, $D \otimes_A A_{\mathfrak{q}} = D_n^i(B \otimes_A A_{\mathfrak{q}} | A_{\mathfrak{q}}, M_B \otimes_A A_{\mathfrak{q}})$. But $\mathfrak{q} \not\supseteq I$ so $B \otimes_A A_{\mathfrak{q}} = A_{\mathfrak{q}}$; therefore $D \otimes_A A_{\mathfrak{q}}$, and so certainly $D_{\mathfrak{p}}$, vanishes, as required.

The second part of the proof is based on the higher Jacobi–Zariski spectral sequence of proposition 1.12. It implies that for each $r \geq 1$ there is a natural, third quadrant, bounded, spectral sequence

$$E_{pq}^1(r) \Longrightarrow D_{p+q}^i(B|k, I^r M_B)$$

which vanishes outside $-i \leq p \leq 0$, whose $-i^{\text{th}}$ column is given by $E^1_{-i,q} = D^i_{q-i}(A|k, I^r M_B)$, and which is described in the range $-i < p \leq 0$ by natural, first quadrant spectral sequences

$$\mathcal{E}^2_{\alpha\beta}(r) = D^{-p}_\alpha(A|k, D^{i+p}_\beta(B|A, I^r M_B)) \Longrightarrow E^1_{p,\alpha+\beta-p}(r).$$

According to the first part of this proof, "$\varprojlim_r$" $\mathcal{E}^2_{\alpha\beta}(r) = 0$, whence the limit of the E-spectral sequences collapses, giving edge map isomorphisms

$$\text{"}\varprojlim_r\text{"}\, D^i_n(A|k, I^r M_B) \xrightarrow{\simeq} \text{"}\varprojlim_r\text{"}\, D^i_n(B|k, I^r M).$$

To complete the proof we only need to check that $I^r M \to I^r M_B$ is an isomorphism for $r \gg 0$. It is surjective since $I \xrightarrow{\simeq} IB$. Let $X = \text{Ker}(M \to M_B)$. For any $\mathfrak{q} \in \text{Spec}\, A$ not containing I, the map $A_{\mathfrak{q}} \to B \otimes_A A_{\mathfrak{q}}$ is an isomorphism and so $X_{\mathfrak{q}} = 0$; since X is a finitely generated A-module, this implies that $I^\lambda X = 0$ for some $\lambda \geq 1$. Next, the usual Artin-Rees lemma implies that $I^r M \cap X \subseteq I^\lambda X = 0$ for $r \gg 0$. For such r, the map $I^r M \to I^r M_B$ is injective. $\qquad\square$

Theorem 2.2 (Pro-excision for AQ homology). *Fix $i \geq 0$ and assume that $i!$ is invertible in A. Then the following square of simplicial pro A-modules is homotopy cartesian*

$$
\begin{array}{ccc}
\mathbb{L}^i_{A|k} \otimes_A M & \longrightarrow & \mathbb{L}^i_{B|k} \otimes_B M_B \\
\downarrow & & \downarrow \\
\text{"}\varprojlim_r\text{"}\, \left(\mathbb{L}^i_{A/I^r|k} \otimes_{A/I^r} M/I^r M\right) & \longrightarrow & \text{"}\varprojlim_r\text{"}\, \left(\mathbb{L}^i_{B/I^r|k} \otimes_{B/I^r} M_B/I^r M_B\right)
\end{array}
$$

resulting in a long exact, Mayer–Vietoris sequence of pro A-modules

$$\cdots \to D^i_n(A|k, M) \to \text{"}\varprojlim_r\text{"}\, D^i_n(A/I^r|k, M/I^r M) \oplus D^i_n(B|k, M_B) \to \text{"}\varprojlim_r\text{"}\, D^i_n(B/I^r|k, M_B/I^r M_B) \to \cdots$$

If moreover $D^i_n(B|k, M_B)$ is an I-adically Artin–Rees B-module for all $n \geq 0$ (e.g., $k \to B$ being v.g.r. and M being finitely generated over A suffices by theorem A.3), then this breaks into short exact sequences:

$$0 \to D^i_n(A|k, M) \to \text{"}\varprojlim_r\text{"}\, D^i_n(A/I^r|k, M/I^r M) \oplus D^i_n(B|k, M_B) \to \text{"}\varprojlim_r\text{"}\, D^i_n(B/I^r|k, M_B/I^r M_B) \to 0.$$

Proof. According to theorem 1.7(ii) we may replace the bottom of the diagram by the arrow

$$\text{"}\varprojlim_r\text{"}\, \left(\mathbb{L}^i_{A|k} \otimes_A M/I^r M\right) \longrightarrow \text{"}\varprojlim_r\text{"}\, \left(\mathbb{L}^i_{B|k} \otimes_B M_B/I^r M_B\right)$$

The vertical arrows in the diagram now become surjective, and it is enough to prove that the induced map on the their kernels, namely

$$\text{"}\varprojlim_r\text{"}\, \left(\mathbb{L}^i_{A|k} \otimes_A I^r M\right) \longrightarrow \text{"}\varprojlim_r\text{"}\, \left(\mathbb{L}^i_{B|k} \otimes_B I^r M_B\right)$$

is a quasi-isomorphism. But this is the content of the previous lemma, completing the proof that the square is homotopy cartesian.

Under the assumption of the 'If moreover' claim, theorem 1.7(ii)&(iii) imply (see also remark 1.10) that

$$D_n^i(B|k, M_B) \longrightarrow \text{``}\varprojlim_r\text{''}\, D_n^i(B/I^r|k, M_B/I^r M_B)$$

is surjective for all $n \geq 0$, which is clearly enough to break up the Mayer–Vietoris sequence. $\square$

For the sake of the completeness, we restate the theorem in the special case $M = A$:

Corollary 2.3. *Fix $i \geq 0$ and assume that $i!$ is invertible in A. Then the following square of simpicial pro A-modules is homotopy cartesian*

$$
\begin{array}{ccc}
\mathbb{L}_{A|k}^i & \longrightarrow & \mathbb{L}_{B|k}^i \\
\downarrow & & \downarrow \\
\text{``}\varprojlim_r\text{''}\, \mathbb{L}_{A/I^r|k}^i & \longrightarrow & \text{``}\varprojlim_r\text{''}\, \mathbb{L}_{B/I^r|k}^i
\end{array}
$$

resulting in a long exact, Mayer–Vietoris sequence of pro A-modules

$$\cdots \to D_n^i(A|k) \to \text{``}\varprojlim_r\text{''}\, D_n^i(A/I^r|k) \oplus D_n^i(B|k) \to \text{``}\varprojlim_r\text{''}\, D_n^i(B/I^r|k) \to \cdots$$

If moreover $D_n^i(B|k)$ is an I-adically Artin–Rees B-module for all $n \geq 0$ (e.g., $k \to B$ being v.g.r. suffices by theorem A.3), then this breaks into short exact sequences:

$$0 \to D_n^i(A|k) \to \text{``}\varprojlim_r\text{''}\, D_n^i(A/I^r|k) \oplus D_n^i(B|k) \to \text{``}\varprojlim_r\text{''}\, D_n^i(B/I^r|k) \to 0$$

3   ARTIN–REES PROPERTIES FOR HOCHSCHILD HOMOLOGY

Given a morphism $k \to A$ of rings, and an A-module M, we let $H_*^{\text{naive},k}(A, M)$ denote the 'usual' Hochschild homology of A as a k-algebra with coefficients in M; it is the homology of the Hochschild complex $C_\bullet^k(A, M)$. In particular, $HH_*^{\text{naive},k}(A) = H_*^{\text{naive},k}(A, A)$ denotes the usual Hochschild homology of A as a k-algebra.

However, it will be more convenient to work with the derived version of Hochschild homology, for which we use the notation $H_*^k(A, M)$. That is, letting $P_\bullet \to A$ be a simplicial resolution of A by free k-algebras, $H_*^k(A, M)$ is defined to be the homology (of the diagonal) of the bisimplicial A-module

$$C_q^k(P_p, M) = M \otimes_k P_p^{\otimes_k q} \qquad\qquad (p, q \geq 0)$$

In the special case $A = M$ we write $HH_*^k(A) = H_*^k(A, M)$.

Lemma 3.1. *Let $k \to A$ be a morphism of rings, and let M be an A-module. Then*

(i) If $k \to A$ is flat (e.g., k a field) then $H_*^k(A, M) \cong H_*^{naive,k}(A, M)$.

(ii) *There is a first quadrant spectral sequence*

$$E_{pq}^2 = D_p^q(A|k, M) \Longrightarrow H_{p+q}^k(A, M).$$

Moreover, if $N!$ is invertible in A then this spectral sequence degenerates in degrees $n \leq N$, yielding

$$H_n^k(A, M) \cong \bigoplus_{p=0}^{n} D_p^{n-p}(A|k, M).$$

(iii) *If k is Noetherian, $k \to A$ is essentially of finite type, and M is finitely generated over A, then $H_n^k(A, M)$ is a finitely generated A-module for all $n \geq 0$.*

Proof. If $k \to A$ is flat then $M \otimes_k P_\bullet^{\otimes_k q} \to M \otimes_k A^{\otimes_k q}$ is a quasi-isomorphism for all $q \geq 0$, and so the bisimplicial A-module $C_\bullet^k(A, P_\bullet)$ is quasi-isomorphic to $C_\bullet^k(A, M)$. This proves (i).

The spectral sequence of (ii) is the usual spectral sequence of the bisimplicial module $C_\bullet^k(P_\bullet, M)$: indeed, fixing p and taking homology in the q-direction gives

$$H_q(C_\bullet^k(P_p, M)) = H_q(P_p|k, M) \cong \Omega_{P_p|k}^q \otimes_{P_p} M,$$

where the isomorphism follows from the fact that P_p is a free k-algebra. Now taking homology in the p-direction gives

$$H_p(\Omega_{P_\bullet|k}^q \otimes_{P_\bullet} M) = D_p^q(A|k, M),$$

which gives the desired spectral sequence. Regarding degeneration of the spectral sequence, the argument in [16, Thm. 3.5.8] for the special case that $\mathbb{Q} \subseteq k$ works in our more general situation.

(iii) follows from (ii) and the analogous result in André–Quillen homology, which was mentioned in our original discussion of that theory. $\square$

Using the direct sum decomposition for $H_n^k(A, M)$, it is immediate that our Artin–Rees results for André–Quillen homology given in theorem 1.7 and corollary 1.8 remain true for Hochschild homology:

Theorem 3.2. *Let $k \to A$ be a ring homomorphism, let I be an ideal of A, let M be an A-module, and fix an integer $n \geq 0$ such that $n!$ is invertible in A. Consider the following statements:*

(i) *"$\varprojlim$"$_r$ $H_n^A(A/I^r, M/I^r M) = 0$.*

(ii) *The natural map*

$$\text{``}\varprojlim_r\text{''}\, H_n^k(A, M/I^r M) \longrightarrow \text{``}\varprojlim_r\text{''}\, H_n^k(A/I^r, M/I^r M)$$

is an isomorphism.

(iii) *The natural map*

$$\text{``}\varprojlim_r\text{''}\, H_n^k(A, M) \otimes_A A/I^r \longrightarrow \text{``}\varprojlim_r\text{''}\, H_n^k(A, M/I^r M)$$

is an isomorphism.

(ii) is true if A/I is I-adically Artin–Rees (e.g., A Noetherian suffices). (iii) is true if $H_n^k(A, M)$ is I-adically Artin-Rees (e.g., $k \to A$ being virtually geometrically regular and M being finitely generated over A suffice, by theorem A.3 and the decomposition above.)

4   PRO-EXCISION FOR HH, HC, AND K-THEORY

We begin by introducing our notation for cyclic homology. Given a morphism $k \to A$ of rings, $HC_*^{\mathrm{naive},k}(A)$ denotes the usual cyclic homology of the k-algebra A, which is the homology of the cyclic bicomplex $CC^k(A)$. Just as for Hochschild homology we prefer to denote by $HC_*^k(A)$ the derived version, defined as the homology of the simplicial bicomplex $CC^k(P_\bullet)$, where $P_\bullet \to A$ is a simplicial resolution of A by free k-algebras.

If $k \to A$ is flat then $HC_*^k(A) \cong HC_*^{\mathrm{naive},k}(A)$. In any case, the SBI sequence remains valid in the derived setting:

$$\cdots \to HH_n^k(A) \to HC_n^k(A) \to HC_{n-2}^k(A) \to \cdots$$

In fact, given that we will often assume $n! \in A^\times$ in our results, $HC_n^{\mathrm{naive},k}(A)$ and $HC_n^k(A)$ may alternatively be defined via Connes cyclic complex (e.g., see [16, Rmk. in §2.1]).

The relative groups of (derived) Hochschild and cyclic homology, with respect to an ideal $I \subseteq A$, are denoted $HH_*^k(A, I)$ and $HC_*^k(A, I)$ as usual; they fit into the long exact sequences

$$\cdots \to HH_n^k(A, I) \to HH_n^k(A) \to HH_n^k(A/I) \to \cdots$$

$$\cdots \to HC_n^k(A, I) \to HC_n^k(A) \to HC_n^k(A/I) \to \cdots$$

Our pro-excision result for André–Quillen homology, namely theorem 2.2, immediately extends to Hochschild and cyclic homology:

Theorem 4.1. *Let $k \to A \to B$ be morphisms of Noetherian rings such that $A \to B$ is essentially of finite type; suppose that I is an ideal of A mapped isomorphically to an ideal of B. Fix an integer $n \geq 0$ such that $n!$ is invertible in A. Then the natural maps*

$$\text{``}\varprojlim_r\text{''}\, HH_n^k(A, I^r) \to \text{``}\varprojlim_r\text{''}\, HH_n^k(B, I^r), \qquad \text{``}\varprojlim_r\text{''}\, HC_n^k(A, I^r) \to \text{``}\varprojlim_r\text{''}\, HC_n^k(B, I^r)$$

are isomorphisms.

Proof. The claim for HH reduces to the analogous claim for $D_p^{(n-p)}$, with $0 \leq p \leq n$; but that is precisely pro-excision for André–Quillen homology, namely corollary 2.3.

The claim for HC follows in the usual way by repeatedly applying the five lemma to "$\varprojlim_r$" of the SBI sequences for the relative groups:

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & HH_n^k(A, I) & \longrightarrow & HC_n^k(A, I) & \longrightarrow & HC_{n-2}^k(A, I) & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & HH_n^k(B, I) & \longrightarrow & HC_n^k(B, I) & \longrightarrow & HC_{n-2}^k(B, I) & \longrightarrow & \cdots
\end{array}
$$

$\square$

The previous theorem readily extends to K-theory, thereby establishing our first main theorem:

Theorem 4.2. *Let $A \to B$ be an essentially finite type morphism of Noetherian $\mathbb{Q}$-algebras, and let I be an ideal of A mapped isomorphically to an ideal of B. Then*

$$\text{``}\varprojlim_r\text{''}\, K_n(A, B, I^r) = 0$$

for all $n \in \mathbb{Z}$; i.e., the natural maps

$$\text{``}\varprojlim_r\text{''}\, K_n(A, I^r) \longrightarrow \text{``}\varprojlim_r\text{''}\, K_n(B, I^r)$$

are all isomorphisms.

Proof. Since negative K-theory satisfies excision we may assume $n \geq 0$ (even $n > 0$). The previous theorem, with $k = \mathbb{Q}$ (so that derived HC and naive HC are the same), implies that $\text{``}\varprojlim_r\text{''}\, HC_{n-1}^{\mathbb{Q}}(A, B, I^r) = 0$. But this is isomorphic to $\text{``}\varprojlim_r\text{''}\, K_n(A, B, I^r)$ by Cortiñas proof of the KABI conjecture [3]. $\square$

Remark 4.3. It is worth pointing our several other consequences:

(i) In the HH isomorphism of theorem 4.1, we may of course insert any finitely generated A-module M. Thus, under the assumptions of theorem 4.1, we obtain a Mayer–Vietoris sequence

$$\cdots \to H_d^k(A, M) \to \text{``}\varprojlim_r\text{''}\, H_d^k(A/I^r, M/I^r M) \oplus H_d^k(B, M_B) \to \text{``}\varprojlim_r\text{''}\, H_d^k(B/I^r, M_B/I^r M_B) \to \cdots$$

in degree $d < n$. If $k \to B$ is v.g.r. then the 'If moreover' part of theorem 2.2 implies that this breaks into short exact sequences. In particular, this means that the natural map

$$\text{Ker}\left(H_d^k(A, M) \to H_d^k(B, M)\right) \longrightarrow H_d^k(A/I^r, M/I^r M)$$

is injective for $r \gg 0$. In other words, the kernel in Hochschild homology when passing from A to B is captured by a sufficiently large quotient of A.

(ii) Since periodic cyclic homology satisfies excision (by the Cuntz–Quillen theorem [6]), it a fortiori satisfies pro-excision; so the fibre sequence $HN \to HP \to HC[2]$ and pro-excision for HC implies that HN also satisfies pro-excision. More precisely, if A, B, I, n are as in the statement of theorem 4.1, then the natural map

$$\text{``}\varprojlim_r\text{''}\, HN_n^k(A, I^r) \longrightarrow \text{``}\varprojlim_r\text{''}\, HN_n^k(B, I^r)$$

is an isomorphism.

(iii) Similarly to (ii), excision for KV-theory (= Karoubi–Villamayor theory) and pro-excision for K-theory implies that $K^{\text{nil}} = \text{hofib}(K \to KV)$ satisfies pro-excision for essentially finite type morphisms of Noetherian $\mathbb{Q}$-algebras.

5   Pro-descent for André–Quillen homology

The second part of the paper, concerning descent in the h-topology rather than excision, now begins.

5.1 André–Quillen homology for schemes

Let k be a ring and fix $i \geq 0$. Here we extend André–Quillen homology from k-algebras to schemes in the standard way using hypercohomology. The analogue for Hochschild/cyclic homology was carried out in [30, 28], and the case of André–Quillen homology is verbatim equivalent; therefore we will offer precise statements but few details.

First notice that if R is a k-algebra then is possible to choose a simplicial resolution $P_\bullet \to R$ by free k-algebras in a way which is functorial in R; e.g., via the comonad associating to R the free k-algebra generated by the set R, see [29, E.g. 8.6.16]. Thus the cotangent complex $\mathbb{L}_{R|k}$ may be chosen to be functorial in R (and not merely up to homotopy). Now let X be a scheme over k, and let $\widetilde{\mathbb{L}}^i_{X|k}$ denote the (chain) complex of sheaves on X obtained by degree-wise sheafifying (in the Zariski topology)

$$U \mapsto \mathbb{L}^i_{\mathcal{O}_X(U)|k}.$$

Let $\widetilde{\mathbb{L}}^i_{X|k,\mathrm{neg}}$ be the (cochain) complex of sheaves supported in degree ≤ 0 obtained from $\widetilde{\mathbb{L}}^i_{X|k}$ by negating its numbering (this is for the sake of (ii) below). Given a quasi-coherent $\mathcal{O}_X$-module M, we let $\mathbb{H}_{\mathrm{Zar}}(X, \widetilde{\mathbb{L}}^i_{X|k,\mathrm{neg}} \otimes_{\mathcal{O}_X} M)$ be the global sections of the total complex of a Carten–Eilenberg resolution of $\widetilde{\mathbb{L}}^i_{X|k,\mathrm{neg}} \otimes_{\mathcal{O}_X} M$, and we define the André–Quillen homology of X, relative to k, with coefficients in M to be its cohomology

$$D^i_*(X|k, M) := \mathbb{H}^{-*}_{\mathrm{Zar}}(X, \widetilde{\mathbb{L}}^i_{X|k,\mathrm{neg}} \otimes_{\mathcal{O}_X} M).$$

We continue the standard notational abuses of omitting i when it is 1 and omitting M when it is $\mathcal{O}_X$. Note (from (i) and (ii) below) that $D^i_n(X|k)$ can be non-zero for n both positive and negative.

The following consequences are either automatic from the construction or follow exactly as for Hochschild homology in [30, 28] (bearing in mind that André–Quillen homology of rings behaves well under localisation):

(i) ($i = 0$) Since $\mathbb{L}^0_{X|k,\mathrm{neg}} \otimes_{\mathcal{O}_X} M \simeq M$, we have

$$D^0_*(X|k, M) \cong H^{-*}_{\mathrm{Zar}}(X, M).$$

(ii) (Agreement on affines) If $X = \mathrm{Spec}\, A$ is an affine k-scheme, then $D^i_*(X|k, M) = D^i_*(A|k, M)$.

(iii) (Descent) If X has finite Krull dimension and we assume U, V are an open cover of X, then there is a long exact, Mayer–Vietoris sequence

$$\cdots \to D^i_n(X|k, M) \to D^i_n(U|k, M|_U) \oplus D^i_n(V|k, M|_V) \to D^i_n(U \cap V|k, M|_{U \cap V}) \to \cdots$$

(iv) (Hypercohomology spectral sequence) If X has finite Krull dimension then there is a bounded fourth quadrant spectral sequence

$$E_2^{pq} = H^p_{\mathrm{Zar}}(X, \mathcal{D}^i_{-q}(-|k, M)) \Longrightarrow D^i_{-p-q}(X|k, M),$$

where $\mathcal{D}^i_{-q}(-|k, M)$ denotes the Zariski sheafification of $U \mapsto D^i_{-q}(U|k, M|_U)$. This shows that $D^i_*(X|k, M) = 0$ for $* < -\dim X$.

Moreover,

(v) If $0 \to M \to N \to P \to 0$ is a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules, then there is a long exact sequence of André–Quillen homology

$$\cdots \to D_n^i(X|k, M) \to D_n^i(X|k, N) \to D_n^i(X|k, P) \to \cdots$$

(Proof: As in the affine case).

(vi) (Higher Jacobi–Zariski) Let $k \to A$ be a homomorphism of rings, let X be a scheme over A of finite Krull dimension, let M be a quasi-coherent $\mathcal{O}_X$-module, and fix $i \geq 0$. Then there is a natural, bounded spectral sequence of A-modules $E_{pq}^1 \Longrightarrow D_{p+q}^i(X|k, M)$ whose columns may be described as follows:

   (a) Suppose $p < -i$ or $p > 0$. Then $E_{pq}^1 = 0$.
   (b) Suppose $-i \leq p \leq 0$. Then the pth column of the E^1-page is given by a bounded spectral sequence

   $$\mathcal{E}_{\alpha\beta}^2 = D_\alpha^{-p}\big(A|k, D_\beta^{i+p}(X|A, M)\big) \Longrightarrow E_{p,\alpha+\beta-p}^1.$$

   (Note: $\mathcal{E}_{\alpha\beta}^2 = 0$ if α or β is $< -\dim X$.)

(Proof: Imitate the proof in the affine case.)

5.2 Artin–Rees properties for AQ homology of schemes

The following are the scheme-theoretic analogues of the parts of theorem 1.7 and corollary 1.8 which will be required to prove pro-h-descent for André–Quillen homology (as earlier, we write $D_n^i(-)' = D_n^i(-) \otimes_{\mathbb{Z}} \mathbb{Z}[1/i!]$):

Proposition 5.1. *Let k be a ring and X a Noetherian, separated scheme over k which has finite Krull dimension; let $\mathcal{I} \subseteq \mathcal{O}_X$ be an ideal sheaf on X and M a quasi-coherent $\mathcal{O}_X$-module. Fix an integer $i \geq 0$. Then*

 (i) No analogue of theorem 1.7(i)

 (ii) The natural map

$$\text{``}\varprojlim_r\text{''} \, D_n^i(X|k, M/\mathcal{I}^r M)' \longrightarrow \text{``}\varprojlim_r\text{''} \, D_n^i(\operatorname{Spec} \mathcal{O}_X/\mathcal{I}^r|k, M/\mathcal{I}^r M)$$

 is an isomorphism for all $n \in \mathbb{Z}$.

 (iii) Suppose that $\mathcal{I}$ is the pull-back of an ideal $I \subseteq k$ and that X is essentially of finite type over k; then the natural map

$$\text{``}\varprojlim_r\text{''} \, D_n^i(X|k, M) \otimes_k k/I^r \longrightarrow \text{``}\varprojlim_r\text{''} \, D_n^i(X|k, M/\mathcal{I}^r M)$$

 is an isomorphism for all $n \in \mathbb{Z}$.

Proof. By descent and an induction on the size of an affine open cover of X, these claims reduce to corollary 1.8(i) and theorem 1.7(ii), (iii) respectively (which are vacuously true in negative degree). $\square$

5.3   Pro-h-descent for AQ homology

Now we are prepared to extend the calculations of section to the case of descent. We begin with a simple finite generation result:

Lemma 5.2. *Let A be a Noetherian ring, and X a proper scheme over A of finite Krull dimension. Then $D^i_n(X|A, M)$ is a finitely generated A-module, for any coherent $\mathcal{O}_X$-module M and any integers $n \in \mathbb{Z}$, $i \geq 0$.*

Proof. Each $\mathcal{O}_X$-module $\mathcal{D}^i_n(-|A, M)$ is coherent since X is of finite type over A. Since X is proper over A, this implies that each of the cohomology groups of $\mathcal{D}^i_n(-|A, M)$ is a finitely generated A-module; so the claim follows at once from the hypercohomology spectral sequence. □

We also need an analogue of lemma 2.1:

Lemma 5.3. *Let $k \to A$ be a morphism of Noetherian rings, and $\pi : X \to A$ a proper morphism, where X has finite Krull dimension; suppose that $I \subseteq A$ is an ideal such that the induced map $X \setminus V(\pi^*I) \to \operatorname{Spec} A \setminus V(I)$ is an isomorphism. Let M be a finitely generated A-module. Then there are natural isomorphisms*

$$\text{``}\varprojlim_r\text{''} \, D^i_n(X|A, \pi^*(I^r M)) \xrightarrow{\simeq} \text{``}\varprojlim_r\text{''} \, I^r D^i_n(X|A, \pi^*M) \overset{(*)}{\cong} 0$$

and

$$\text{``}\varprojlim_r\text{''} \, D^i_n(A|k, I^r M) \xrightarrow{\simeq} \text{``}\varprojlim_r\text{''} \, D^i_n(X|k, \pi^*(I^r M))$$

for all $n \in \mathbb{Z}$ and $i \geq 0$ (except that isomorphism $(*)$ is not valid if $i = n = 0$).

Proof. Just as in the proof of lemma 2.1, the first isomorphism follows from the long exact homology sequence for $0 \to \mathcal{I}^r M \to M \to M/\mathcal{I}^r M \to 0$ and proposition 5.1(iii). The vanishing claim follows from the fact that $D^i_n(X|A, M)$ is a finitely generated A-module (by the previous lemma) which is supported on I (except when $n = i = 0$).

The final isomorphism is also proved just as in lemma 2.1, this time using the scheme version of the Jacobi–Zariski sequence. □

We are finally equipped to prove that André–Quillen homology satisfies 'pro-descent' with respect to the h-topology. Recall that an *abstract blow-up square* of schemes is a pull-back diagram

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

where $X' \to X$ is proper, $Y \to X$ is a closed embedding, and the induced map $X' \setminus Y' \to X \setminus Y$ is an isomorphism. Morever, given a closed subscheme $Y \hookrightarrow X$, we denote by rY the fattened subscheme given by $rY = V(\mathcal{I}^r) = \operatorname{Spec} \mathcal{O}_X/\mathcal{I}^r$, where $\mathcal{I} \subseteq \mathcal{O}_X$ is the sheaf of ideals defining Y. The following theorem is true more generally with a general coherent module M in place of $\mathcal{O}_X$, but we omit it to simplify the notation:

Theorem 5.4 (Pro-*h*-descent for AQ homology). *Let k be a Noetherian ring, and let*

$$
\begin{array}{ccc}
Y' & \xrightarrow{\ j'\ } & X' \\
{\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\
Y & \xrightarrow{\ j\ } & X
\end{array}
$$

be an abstract blow-up square of schemes over k. Assume that the schemes are separated, Noetherian, and have finite Krull dimension. Fix an integer $i \geq 0$ such that $i!$ is invertible in k.

Then the following square of complexes of pro k-modules is homotopy cartesian

$$
\begin{array}{ccc}
\mathbb{H}_{Zar}(X, \widetilde{\mathbb{L}}^i_{X|k,neg}) & \longrightarrow & \mathbb{H}_{Zar}(X', \widetilde{\mathbb{L}}^i_{X'|k,neg}) \\
\downarrow & & \downarrow \\
\text{``}\varprojlim\text{''}_r \mathbb{H}_{Zar}(rY, \widetilde{\mathbb{L}}^i_{rY|k,neg}) & \longrightarrow & \text{``}\varprojlim\text{''}_r \mathbb{H}_{Zar}(rY', \widetilde{\mathbb{L}}^i_{rY'|k,neg})
\end{array}
$$

resulting in a long exact, Mayer–Vietoris sequence of pro k-modules

$$
\cdots \to D^i_n(X|k) \to \text{``}\varprojlim_r\text{''}\, D^i_n(rY|k) \oplus D^i_n(X'|k) \to \text{``}\varprojlim_r\text{''}\, D^i_n(rY'|k) \to \cdots
$$

Proof. The proof is very similar to that of theorem 2.2. Firstly, by descent and an induction on the size of an affine open cover of X we may assume that $X = \operatorname{Spec} A$ is affine; let $Y = \operatorname{Spec} A/I$ for some ideal $I \subseteq A$.

Next, according to proposition 5.1(ii), we may replace the bottom of the diagram by $\mathbb{H}_{\mathrm{Zar}}$ of the map

$$
\text{``}\varprojlim_r\text{''}\, \widetilde{\mathbb{L}}^i_{X|k} \otimes_{\mathcal{O}_X} \mathcal{O}_X/I^r \to \text{``}\varprojlim_r\text{''}\, \widetilde{\mathbb{L}}^i_{X'|k} \otimes_{\mathcal{O}_{X'}} \mathcal{O}'_X/\pi^* I^r
$$

The vertical arrows in the diagram are now induced by surjections between the respective cotangent complexes, and it is enough to prove that the induced map on their kernels, namely $\mathbb{H}_{\mathrm{Zar}}$ of

$$
\text{``}\varprojlim_r\text{''}\, \widetilde{\mathbb{L}}^i_{X|k} \otimes_{\mathcal{O}_X} I^r \to \text{``}\varprojlim_r\text{''}\, \widetilde{\mathbb{L}}^i_{X'|k} \otimes_{\mathcal{O}_{X'}} \pi^* I^r
$$

is a quasi-isomorphism. But this iss precisely the previous lemma when $M = A$. $\qquad\square$

6   PRO-H-DESCENT FOR HH, HC, AND K-THEORY

In this section we extend pro-h-descent for André–Quillen homology to Hochschild and cylic homology and K-theory. This is done exactly as for excision in section 4. With the target in mind being pro-h-descent for K-theory, we will not prove the best possible results for HH and HC, but focus on the required case for K-theory; in particular, our base will be a field, eliminating the need of introducing a derived version of HH and HC for schemes.

Given a field k and a scheme X over k, we denote by $HH^k_*(X)$ and $HC^k_*(X)$, where $* \in \mathbb{Z}$, the Hochschild and cyclic homology of X over k, as formulated in, e.g. [30, 28]. If X

has finite Krull dimension then there is a bounded spectral sequence $E_{pq}^2 = D_p^q(X|k, M) \Longrightarrow HH_{p+q}^k(X)$, and if $k \supseteq \mathbb{Q}$ this degenerates to a finite direct sum decomposition

$$HH_n^k(X) = \bigoplus_{p \in \mathbb{Z}} D_p^{n-p}(X|k)$$

for each $n \in \mathbb{Z}$ (notice that the D_p^{n-p} summand is zero outside the range $-\dim X \le p \le n$, and so in particular $HH_n^k(X) = 0$ if $n < -\dim X$).

Given a closed embedding of schemes $Y \hookrightarrow X$ over the field k, we will write $HH_*^k(X, Y)$, $HC_*^k(X, Y)$ for the relative groups.

Theorem 6.1. *Let k be a field containing $\mathbb{Q}$, and let*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

be an abstract blow-up square of schemes over k, all of which are separated and have finite Krull dimension.

Then the natural maps

$$\text{``}\varprojlim_r\text{''}\, HH_n^k(X, rY) \to \text{``}\varprojlim_r\text{''}\, HH_n^k(X', rY'), \qquad \text{``}\varprojlim_r\text{''}\, HC_n^k(X, rY) \to \text{``}\varprojlim_r\text{''}\, HC_n^k(X', rY')$$

are isomorphisms for all $n \in \mathbb{Z}$.

Proof. The claim for HH follows from the analogous claim as for André–Quillen homology, namely theorem 5.4, via the direct sum decomposition above. To pass to HC, use the five lemma and induction up the limit of the SBI sequences

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & HH_n^k(X, rY) & \longrightarrow & HC_n^k(X, rY) & \longrightarrow & HC_{n-2}^k(X, rY) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & HH_n^k(X', rY') & \longrightarrow & HC_n^k(X', rY') & \longrightarrow & HC_{n-2}^k(X, rY) & \longrightarrow & \cdots \end{array}$$

(notice it is possible to start the induction since the SBI sequences vanishes in degrees $< -\max\{\dim X, \dim X'\}$). $\square$

From this we may establish pro-h-descent for K-theory:

Theorem 6.2. *Let k be a field of characteristic zero, and let*

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

be an abstract blow-up square of schemes which are separated and of finite type over k. Then the natural map

$$\text{``}\varprojlim_r\text{''}\, K_n(X, rY) \longrightarrow \text{``}\varprojlim_r\text{''}\, K_n(X', rY')$$

is an isomorphism for all $n \in \mathbb{Z}$.

Proof. For any scheme Z, we let $HN(Z)$ and $HP(Z)$ denote its negative and periodic cyclic homologies over $\mathbb{Q}$, as formulated in [4]. Moreover, the infinitesimal K-theory of Z is defined to be the homotopy fibre of the Chern character from K-theory to negative cyclic homology, i.e.

$$K^{\mathrm{inf}}(Z) = \mathrm{hofib}(K(Z) \to NH(Z)).$$

According to [Thm. 4.6, loc. cit.] and [Cor. 3.13, loc. cit.], K^{inf} and HP satisfy descent with respect to the h-topology, so a fortiori satisfy pro-h-descent. The fibre sequences

$$K^{\mathrm{inf}} \to K \to NH, \qquad HN \to HP \to HC[2]$$

therefore imply that pro-h-descent for HC (the previous theorem) implies it for NH and for K-theory.

This completes the proof of our second main theorem. $\qquad\qquad\qquad\qquad\square$

Remark 6.3. Notice that if $X' \to X$ is a finite morphism, then pro-h-descent can be readily deduced from pro-excision.

A   APPENDIX

The appendix is dedicated to two topics. First we discuss 'virtually geometrically regular' morphisms and show that they provide a manageable test for the Andr–Quillen or Hoschschild homology groups to be I-adically Artin–Rees. Secondly we show how theorem 1.7(ii) easily yields pro versions of the Hochschild–Konstant–Rosenberg theorem.

A.1   An example when the André-Quillen groups are Artin-Rees

The validity of theorem 1.7(iii) depended upon knowing that the André-Quillen homology groups were I-adically Artin-Rees. This is satisfied if k is Noetherian, $k \to A$ is essentially of finite type, and M is finitely generated over A, but those conditions are too restrictive for applications: for example, we would like to let $k = \mathbb{Q}$ while A is of finite type over an arbitrary characteristic zero field. For the sake of convenience, we make a definition:

Definition A.1. We will say that a morphism of Noetherian rings $k \to A$ is *virtually geometrically regular* (or that A is a virtually geometrically regular k-algebra) if and only if the morphism admits a factorisation $k \to K \to A$, where $k \to K$ is geometrically regular and faithfully flat, and $K \to A$ is essentially of finite-type. (We only speak of geometrically regular morphisms when both rings are Noetherian, so K is implicitly Noetherian here.)

Example A.2. There are two important examples:

(i) An essentially finite type morphism between Noetherian rings is obviously v.g.r.

(ii) Let A be a Noetherian ring which is essentially of finite type over some subfield K. Let $k \subseteq K$ be the prime subfield (i.e., $\mathbb{Q}$ or $\mathbb{F}_p$). Then the factorisation $k \to K \to A$ shows that A is a v.g.r. k-algebra.

The following theorem shows that theorem 1.7(iii) is valid for v.g.r. morphisms:

Theorem A.3. *Let $k \to A$ be a virtually geometrically regular morphism of Noetherian rings. Then $D_n^i(A|k, M)$ is I-adically Artin-Rees for any ideal $I \subseteq A$, any finitely generated A-module M, and all $i, n \geq 0$.*

Remark A.4. More precisely, the proof will show that $D_n^i(A|k)$ becomes, after a faithfully flat base change, a finite direct sum of modules of the form

$$\text{finitely generated module} \otimes \text{free module}.$$

The Artin-Rees claim will easily follow from this description.

Before the proof, we need both the Künneth formula for André-Quillen homology (c.f., [11]) and a consequence of it:

Lemma A.5. *Let k be a ring, let A be a k algebra, let k' be a flat k-algebra, and put $A' = A \otimes_k k'$. Let $i \geq 0$.*

(i) *There is a natural decomposition of simplicial A'-modules*

$$\mathbb{L}_{A'|k}^i \cong \bigoplus_{j=0}^i \mathbb{L}_{A|k}^j \otimes_k \mathbb{L}_{k'|k}^{i-j}$$

(ii) *Assume further that $D_n^{i-j}(k'|k)$ is flat over k' for $j = 0, \ldots, i$ and $n \geq 0$. Let M be an A'-module. Then there is a natural isomorphism of graded A'-modules*

$$D_*^i(A'|k, M) \cong \bigoplus_{j=0}^i D_*^j(A|k, M) \otimes_{k'} D_*^{i-j}(k'|k).$$

If $M = N \otimes_A A'$ for some A-module N, then the terms on the right simplify to $D_*^j(A|k, N) \otimes_k D_*^{i-j}(k'|k)$.

Proof. We begin with a couple of observations on free k-algebras. Let V, W be free k-modules, and let $P = \operatorname{Sym} V$, $Q = \operatorname{Sym} W$ be the associated free k-algebras. Then $R := P \otimes_k Q = \operatorname{Sym}(V \oplus W)$ is also a free k-algebra. Moreover, the Kähler differentials of a free k-algebra are described as follows:

$$P \otimes_k V \xrightarrow{\cong} \Omega_{P|k}^1, \quad \alpha \otimes v \mapsto \alpha \, dv$$

It follows that the natural map

$$(\Omega_{P|k}^1 \otimes_P R) \oplus (\Omega_{Q|k}^1 \otimes_Q R) \to \Omega_{R|k}^1$$

is an isomorphism of R-modules.

Now let $P_\bullet \to A$ and $Q_\bullet \to k'$ be simplicial resolutions by free k-algebras; put $R_\bullet := P_\bullet \otimes_k Q_\bullet$. From the flatness of $k \to k'$ it follows that $R_\bullet \simeq A'$, and so the first of the observations above implies that $R_\bullet$ is a simplicial resolution of A' by free k-algebras. The second observation above implies that the natural map

$$(\Omega_{P_\bullet|k}^1 \otimes_{P_\bullet} R_\bullet) \oplus (\Omega_{Q_\bullet|k}^1 \otimes_{Q_\bullet} R_\bullet) \to \Omega_{R_\bullet|k}^1$$

is an isomorphism of $R_\bullet$-modules, and so after applying $-\otimes_{P_\bullet} A'$ we deduce that we have a natural isomorphism of cotangent complexes:

$$(\mathbb{L}_{A|k}\otimes_A A')\oplus(\mathbb{L}_{k'|k}\otimes_{k'} A')\xrightarrow{\simeq}\mathbb{L}_{A'|k}$$

Taking the ith exterior power over A' yields the required natural decomposition to prove (i):

$$\mathbb{L}^i_{A'|k}\cong\bigoplus_{j=0}^i(\mathbb{L}^j_{A|k}\otimes_A A')\otimes_{A'}(\mathbb{L}^{i-j}_{k'|k}\otimes_{k'} A')$$
$$\cong\bigoplus_{j=0}^i\mathbb{L}^j_{A|k}\otimes_k\mathbb{L}^{i-j}_{k'|k}$$

Claim (ii) follows by tensoring by M and applying the usual Künneth formula for complexes; the special case $M=N\otimes_A A'$ is a consequence of flat base change. $\square$

Corollary A.6. *Let k be a Noetherian ring, let A be a k-algebra, and let k' be a k-algebra which can be written as a filtered inductive limit of finite-type, smooth k-algebras; put $A'=A\otimes_k k'$. Let M be an A'-algebra. Then there is a natural isomorphism of A'-modules*

$$D_n^i(A'|k,M)\cong\bigoplus_{j=0}^i D_n^j(A|k,M)\otimes_{k'}\Omega^{i-j}_{k'|k}$$

for any $n,i\geq 0$.
If $M=N\otimes_A A'$ for some A-module N, then the terms on the right simplify to $D_n^j(A|k,N)\otimes_k\Omega^{i-j}_{k'|k}$.

Proof. The decomposition is an immediate consequence of the previous lemma, once one makes two observations: firstly, the HKR theorem implies that

$$D_n^i(k'|k)=\begin{cases}\Omega^i_{k'|k}&n=0\\0&\text{else}\end{cases}$$

and, secondly, $\Omega^i_{k'|k}$ is a filtered inductive limit of free k' algebras, hence is flat. $\square$

Now we may prove theorem A.3:

Proof. Since $k\to A$ is v.g.r. it admits a factorisation $k\to K\to A$, where K is Noetherian, $k\to K$ is geometrically regular and faithfully flat, and $K\to A$ is essentially of finite-type. Set $A_K=A\otimes_k K$, $K_K=K\otimes_k K$, and $M_K=M\otimes_k K$. Then $A_K=A\otimes_K K_K$ and K_K is a filtered inductive limit of finite-type, smooth K-algebras (since the same is true of $k\to K$ by Neron–Popescu desingularisation [19, 20]). The previous corollary therefore implies that

$$D_n^i(A_K|K,M_K)\cong\bigoplus_{j=0}^i D_n^j(A|K,M)\otimes_K\Omega^{i-j}_{K_K|K}.$$

Notice that the right terms may be simplified to $D_n^{(j)}(A|K,M)\otimes_k\Omega^{i-j}_{K|k}$ and that this is a finitely-generated A-module tensored by a flat K-module.

However, André-Quillen homology commutes with flat base change, so

$$D_n^i(A_K|K, M_K) = D_n^i(A|k, M) \otimes_A A_K.$$

In conclusion, we have written

$$D_n^i(A|k, M) \otimes_A A_K \cong \left(\bigoplus_{j=0}^i E_j \otimes_A F_j \right) \otimes_A A_K,$$

where $E_j := D_n^j(A|K, M)$ are finitely generated A-modules and $F_j := \Omega_{K|k}^{i-j} \otimes_K A$ are flat A-modules. So, for any $m, r \geq 1$ we have

$$\operatorname{Tor}_m^A(D_n^i(A|k), A/I^r) \otimes_A A_K \cong \bigoplus_{j=0}^i \operatorname{Tor}_m^A(E_j, A/I^r) \otimes_A F_j \otimes_A A_K$$

But each E_j is I-adically Artin-Rees (see remark 1.6) so there is $s \geq r$ such that the map $\operatorname{Tor}_m^A(E_j, A/I^s) \to \operatorname{Tor}_m^A(E_j, A/I^r)$ is zero for all j; hence $\operatorname{Tor}_m^A(D_n^i(A|k), A/I^s) \to \operatorname{Tor}_m^A(D_n^i(A|k), A/I^r)$ is zero (recall that $A \to A_K$ is faithfully flat), completing the proof. $\qquad \square$

A.2   Pro Hochschild–Konstant–Rosenberg theorems

Versions of the following 'pro Hochschild-Kostant-Rosenberg theorems', when the base is a field, are explicitly contained in [5] (in the language of Hochschild homology) and can also be found in [13]:

Theorem A.7. *Let $k \to A$ be a geometrically regular morphism of Noetherian rings, let $i \geq 0$ be such that $i!$ is invertible in A, let I be an ideal of A, and let M be an A-module. Then*

$$\text{``}\varprojlim_r\text{''} \, D_n^i(A/I^r|k, M/I^r M) = \begin{cases} \text{``}\varprojlim_r\text{''} \, \Omega_{A/I^r|k}^i \otimes_A M & n = 0, \\ 0 & n > 0. \end{cases}$$

Proof. According to theorem 1.7(ii), we may replace the left side by "$\varprojlim_r$" $D_n^i(A|k, M/I^r M)$. Our assumption on A implies that

$$D_n^i(A|k, M/I^r M) = \begin{cases} \Omega_{A|k}^i \otimes_A M/I^r M & n = 0 \\ 0 & n > 0, \end{cases}$$

so to complete the proof we must show that the natural map

$$\text{``}\varprojlim_r\text{''} \, \Omega_{A|k}^i \otimes_A M/I^r M \to \text{``}\varprojlim_r\text{''} \, \Omega_{A/I^r|k}^i \otimes_A M$$

is an isomorphism. Well, from the exact sequence

$$I^r/I^{2r} \to \Omega_{A|k}^1 \otimes_A A/I^r \to \Omega_{A/I^r|k}^1 \to 0$$

one obtains an exact sequence

$$I^r/I^{2r} \otimes_A \Omega_{A|k}^{i-1} \otimes_A M/I^r M \to \Omega_{A|k}^i \otimes_A M/I^r M \to \Omega_{A/I^r|k}^i \otimes_A M \to 0.$$

Since "$\varprojlim_r$" $I^r/I^{2r} = 0$, the desired isomorphism follows by taking the limit. $\qquad \square$

Corollary A.8. *Let $k \to A$, i, I be as in the previous theorem. Then the natural maps* $\Omega^i_{A/I^r|k} \to HH^k_i(A/I^r)$ *induce an isomorphism of pro A-modules*

$$\text{``}\varprojlim_r\text{''} \, \Omega^i_{A/I^r|k} \xrightarrow{\cong} \text{``}\varprojlim_r\text{''} \, HH^k_i(A/I^r).$$

Meanwhile, in cyclic homology,

$$\text{``}\varprojlim_r\text{''} \, HC^k_i(A/I^r) \cong \bigoplus_{p \geq 0} \text{``}\varprojlim_r\text{''} \, H^{i-2p}_{dR}(A/I^r|k).$$

Proof. The claim for Hochschild homology follows from the decomposition of HH_i into André–Quillen homology groups, and the previous proposition with $M = A$. The cyclic homology claim follows from the Hochschild homology claim in the usual way using the five lemma and SBI sequence. $\qquad\square$

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